Lecture 15: Section 6.1. Let $T\colon E \to E$ be an operator on a vector space E, such that E is complex if any of the eigenvalues of T are complex. We have seen that we can write E as a direct sum $E = E_1 \oplus \cdots E_r$, of the generalized eigenspaces $E_k = \operatorname{Ker}(T - \lambda_k I)^{n_k}$, where n_k is the multiplicity of the root of the characteristic polynomial: $p(t) = (t-\lambda_1)^{n_1} \cdots (t-\lambda_r)^{n_r}$.

Suppose that there is only one eigenvalue λ, of multiplicity $n = \dim E$. Set $N = T - \lambda I$ and $S = \lambda I$. Then by the theorem N is nilpotent; $N^n = 0$, for $E = \operatorname{Ker} T^n$, and $SN = NS$, since S is a multiple of the identity. Moreover S is diagonal (in every basis). We can therefore compute $e^T = e^S e^N = e^\lambda I\big(I + N + ... + N^{n-1}/(n-1)!\big)$.

Ex. If $T = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ then $p(t) = (t-2)^2$. Let $S = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $N = T - S = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$.

Since $N^2 = 0$ and $SN = NS$ we have $e^{tT} = e^{tS} e^{tN} = e^{2t}(I + tN) = e^{2t} \begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}$.

In general let $T_k = T\big|_{E_k}$. Since T_k has only the eigenvalue λ_k the previous argument gives that $T_k = S_k + N_k$, where $S_k = \lambda_k I$ on E_k and $N_k = T_k - S_k$ is nilpotent of order n_k. Thus $T = S + N$, where $S = S_1 \oplus \cdots \oplus S_r$ and $N = N_1 \oplus \cdots \oplus N_r$. S is diagonalized by a basis for E that is made up of any bases for the generalized eigenspaces.

Theorem 2. *$T = S + N$, where $SN = NS$ and S is diagonalizable and N is nilpotent. Furthermore the decomposition is unique.*

Ex. If $T_0 = \begin{bmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ then $p(t) = (t+1)^2(t-1)$. The eigenvalue 1 is simple and to find an eigenvector we solve $(T_0 - I)x = 0$ which has a solution $\hat{e}_3 = (0,2,1)$. The eigenvalue -1 is double and we solve $(T_0 + I)x = 0$, or $\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, to find that there is only one linearly independent solution $\hat{e}_1 = (1,0,0)$. By Theorem 1 we know that the dimension of the generalized eigenspace is two so there is another linearly independent solution of $(T+I)^2 x = 0$. Instead of calculating the square we find it by solving $(T_0 + I)x = \hat{e}_1$, or $\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, which has the solution $\hat{e}_2 = (0,1,0)$ e.g.. Let $T = S + N$ be the decomposition in Theorem 2. In the basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ S takes the form: $S_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. If $\hat{x}$ are the coordinates in the basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ and x are the coordinates in the standard basis then $x = Q\hat{x}$ where $Q = \big[\hat{e}_1, \hat{e}_2 \hat{e}_3\big] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and $Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. In the standard basis the matrix for S is $S_0 = QS_1Q^{-1} = ... = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$. Let $N_0 = T_0 - S_0 = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Since $S_0 N_0 = N_0 S_0$ and $N_0^2 = 0$ we have $e^{Tt_0} = e^{tS_0} e^{tN_0} = Qe^{tS_1}Q^{-1}(I + tN_0)$, where $e^{tS_1} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^t \end{bmatrix}$ so $e^{tT_0} = \begin{bmatrix} e^{-t} & te^{-t} & -2te^{-t} \\ 0 & e^{-t} & -2te^{-t} + 2te^t \\ 0 & 0 & e^t \end{bmatrix}$.