Lecture 16: Section 6.2. In this section T will be a real operator and T_C will denote its complexification. If T_C is diagonalizable we say that T is semisimple.

Theorem 1. *We can write $T = S + N$, where S and N are unique operators such that S is semisimple and N is nilpotent and they commute $SN = NS$.*

Proof. By section 6.1 there are unique complex operator S_0 and N_0 such that $T_C = S_0 + N_0$, S_0 is diagonalizable, N_0 is nilpotent and $S_0 N_0 = N_0 S_0$. The theorem follows from proving that S_0 and N_0 are complexifications of real operators. By section 4.1 this is the case if they commute with the conjugation σ. Let $S_1 = \sigma S_0 \sigma^{-1}$ and $N_1 = \sigma N_0 \sigma^{-1}$. Then it is easy to see that S_1 is diagonalizable, N_1 is nilpotent and $S_1 N_1 = N_1 S_1$. We have $T_C = \sigma T_C \sigma^{-1} = \sigma(S_0 + N_0)\sigma^{-1} = S_1 + N_1$, but since S_0 and N_0 are unique such operators it follows that $S_1 = S_0$ and $N_1 = N_0$ and hence $S_0 = S_C$ and $N_0 = N_C$ are complexifications of some unique real operators S and N. It is easy to see that $SN = NS$ and that N is nilpotent, using that the map $A \to A_C$ is one-to-one, e.g. $(SN - NS)_C = S_0 N_0 - N_0 S_0 = 0$ so $SN - NS = 0$. $\square$

It follows that in a suitable basis (left out elements away from the diagonal means 0)

$$S = \begin{bmatrix} \lambda_1 & & & & & & & & \\ & \cdot & & & & & & & \\ & & \cdot & & & & & & \\ & & & \lambda_r & & & & & \\ & & & & a_1 & -b_1 & & & \\ & & & & b_1 & a_1 & & & \\ & & & & & & \cdot & & \\ & & & & & & & \cdot & \\ & & & & & & & & a_s & -b_s \\ & & & & & & & & b_s & a_s \end{bmatrix}$$

Ex If $T_0 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 \end{bmatrix}$ then $p(t) = (t-i)^2(t+i)^2$ and the generalized i-eigenspace in $\mathbf{C}^4$ is given by solving $(T_0 - iI)^2 z = 0$ or $\begin{bmatrix} -2 & 2i & 0 & 0 \\ -2i & -2 & 0 & 0 \\ -2 & 0 & -2 & 2i \\ -4i & -2 & -2i & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, i.e. $z_1 = iz_2$ and $-z_3 + iz_4 = iz_2$ and a complex basis is $u = (i, 1, 0, 1)$ and $v = (i, 1, -i, 0)$. From these we get the real and imaginary parts $\hat{e}_1 = (1, 0, 0, 0)$, $\hat{e}_2 = (0, 1, 0, 1)$, $\hat{e}_3 = (1, 0, -1, 0)$, $\hat{e}_4 = (0, 1, 0, 0)$. In the basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$ the matrix for S is $S_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (Interchanging $\hat{e}_1$ and $\hat{e}_2$ changes the sign of the first two columns.) If $\hat{x}$ are the coordinates in the basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$ and x the coordinates in the standard basis then $x = Q\hat{x}$ where $Q = \big[\hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_4\big]$. S in the standard basis is $S_0 = QS_1Q^{-1} = \dots = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ and $N_0 = T_0 - S_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Then

$$e^{tT_0} = e^{tN_0 + tS_0} e^{tN_0} e^{tS_0} = (I + tN_0)Q^{-1} e^{tS_1} Q, \; e^{tS_1} = \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix}.$$

Section 6.3: Nilpotent Canonical Forms. We have seen that we can write $T = S + N$ where S is semisimple, N is nilpotent and S and N commute. The decmoposition into these two operators is unique but this does not mean that the matrix representations of S and N are unique. In fact given any matrix representation, any similarity transformation of it is another represenation of the same operator. However, in the previous section we saw a particular representation of the semisimple part, the canonical form, which in the complex case is just a diagonal matrix with the roots of the characteristic polynomial repeated with their multiplicity in the diagonal. In this section we will give a particular representation of the nilpotent part. We hence obtain a particular representation operator called the Jordan canonical form. An elementary nilpotent block is

$$\begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & 1 & 0 \end{bmatrix} \tag{6.3.1}$$

with 1's just below the diagonal and 0's elsewhere. We include $[0]$. If $\hat{e}_1, ..., \hat{e}_n$ is the basis in which N has the matrix (6.3.1) then N is nilpotent since

$$N(\hat{e}_1) = \hat{e}_2, \quad N(\hat{e}_2) = \hat{e}_3, \quad \ldots \quad , \quad N(\hat{e}_{n-1}) = \hat{e}_n, \quad N(\hat{e}_n) = 0 \tag{6.3.2}$$

Theorem. *If N is a nilpotent operator then there is a basis giving N a matrix:*

$$A = \mathrm{diag}\{A_1, ..., A_r\}, \tag{6.3.3}$$

where A_j are elementary nilpotent block, listed in order of decreasing size.

We call the matrix in (6.3.3) the *canonical form* of N. The nullspace of each block is one dimensional so the number of blocks is the same as the dimension of the nullspace: $r = \dim \mathrm{Ker}\, A$. The question is how to calculate the canonical form. Let

$$\nu_k = \text{number of elementary } k \times k \text{ blocks} \quad \text{and} \quad \delta_k = \dim \mathrm{Ker}\, N^k; \quad \delta_0 = 0 \tag{6.3.4}$$

Then we have just argued that $\delta_1 = \nu_1 + ... + \nu_n$. Similarly, the nullspace of an elementray block squared is two dimensional, provied that the size of the block $k \geq 2$, if $k = 1$ then it is one dimensional. Hence $\delta_2 = \nu_1 + 2(\nu_2 + ... + \nu_n)$. In the same way $\delta_3 = \nu_1 + 2\nu_2 + 3(\nu_3 + ... + \nu_n)$ and $\delta_k = \nu_1 + 2\nu_2 + ... + k(\nu_k + ... + \nu_n)$. It follows that $\delta_{k+1} - \delta_k = \nu_{k+1} + ... + \nu_n$ and hence:

Theorem. *We have $\nu_k = -\delta_{k-1} + 2\delta_k - \delta_{k+1}$, if we set $\delta_0 = 0$, $\delta_k = \delta_n$, for $k > n$.*

From just knowing $\dim N^k$; $k \geq 1$ the theorem tells us what the canonical form is. From the proof in the appendix one can construct an algorithm for finding a basis that puts the operator in the canonical form.