**Lecture 18: 7.1 Sinks and Sources.** Consider the system

$$x' = Ax \tag{7.1.1}$$

where $A$ is a real linear operator.

We say that the origin $0 \in \mathbf{R}^n$ is a *sink* for the system if all eigenvalues of $A$ have negative real part in which case we also say that $e^{tA}$ is a *contraction*. We have seen that in this case all solutions approach 0 as $t \to \infty$ (*asymptotic stability*).

We say that the origin $0 \in \mathbf{R}^n$ is a *source* for the system if all eigenvalues of $A$ have positive real part in which case we also say that $e^{tA}$ is an *expansion*. In this case all solutions approach $\infty$ as $t \to \infty$.

**Theorem.** *Let $A$ be and $n \times n$ real linear operator with eigenvalues $\lambda_k = a_k + ib_k$, $k = 1, ..., n$ and let $\alpha$ and $\beta$ be real numbers such that $\alpha < a_k < \beta$, for $k = 1, ..., n$. Then there is a constant $K > 0$ such that any solution of $x' = Ax$ satisfies*

$$K^{-1}|x(0)|e^{\alpha t} \le |x(t)| \le K|x(0)|e^{\beta t}, \qquad t \ge 0. \tag{7.1.2}$$

*where $|x| = (x_1^2 + ... + x_n^2)^{1/2}$.*

If $A$ is diagonal with real eigenvalues then $\langle x, Ax\rangle = \lambda_1 x_1^2 + ... + \lambda_n x_n^2$ so

$$\alpha|x|^2 \le \langle x, Ax\rangle \le \beta|x|^2, \tag{7.1.3}$$

(If $A$ is symmetric then it can be diagonalized by an orthogonal change of coordinates $A = Q\hat{A}Q^{-1}$, where $\hat{A} = \text{diag}\{\lambda_1, ..., \lambda_n\}$ and $Q^{-1} = Q^t$. Since $|Q^t x|^2 = \langle Q^t x, Q^t x\rangle = \langle x, QQ^t x\rangle = \langle x, x\rangle = |x|^2$ and $\langle x, Ax\rangle = \langle x, Q\hat{A}Q^t x\rangle = \langle Q^t x, \hat{A}Q^t x\rangle$, (7.1.3) for a symmetric operator follows from (7.1.3) for the diagonal operator $\hat{A}$.) Suppose that $A = \text{diag}\{\lambda_1, ..., \lambda_p, A_{p+1}, ..., A_m\}$ where $\lambda_1, ..., \lambda_p$ are the real eigenvalues and for $k \ge p+1$ $A_k = \begin{bmatrix} a_k & -b_k \\ b_k & a_k \end{bmatrix}$ are $2 \times 2$ blocks corresponding to the complex eigenvalues $a_k + ib_k$ and $a_k - ib_k$. If $y = (y_1, y_2)$ then $\langle y, A_k y\rangle = a_k y_1^2 + a_k y_2^2$. Hence also in this case $\langle x, Ax\rangle = a_1 x_1^2 + .... + a_n x_n^2$, if $a_i = \text{Re } \lambda_i$ so (7.1.3) holds.

Assuming that (7.1.3) hold let us show (7.1.2). We have

$$\frac{d}{dt}|x| = \frac{d}{dt}(x_1^2 + ... + x_n^2)^{1/2} = \frac{x_1 x_1' + ... + x_n x_n'}{(x_1^2 + ... + x_n^2)^{1/2}} = \frac{\langle x, x'\rangle}{|x|} = \frac{\langle x, Ax\rangle}{|x|} \tag{7.1.4}$$

and hence

$$\alpha|x| \le \frac{d}{dt}|x| \le \beta|x| \tag{7.1.5}$$

Let us first prove that $|x(t)| \le e^{\beta t}|x(0)|$ and the inequality $e^{\alpha t}|x(0)| \le |x(t)|$ follows in the same way. Multiplying by the second inequality in (7.1.5) by the integrating factor $e^{-\beta t}$ we get

$$\frac{d}{dt}\Big(e^{-\beta t}|x(t)|\Big) = e^{-\beta t}\frac{d}{dt}|x(t)| - \beta e^{-\beta t}|x(t)| \le 0 \tag{7.1.6}$$

Integrating this from 0 to $t$ gives

$$e^{-\beta t}|x(t)| - |x(0)| \le 0 \tag{7.1.7}$$

and hence

$$|x(t)| \leq e^{\beta t}|x(0)| \tag{7.1.8}$$

Similarly

$$e^{\alpha t}|x(0)| \leq |x(t)| \tag{7.1.8}$$

This proves (7.1.2) for case of a symmetric matrix $A$ or a semisimple matrix already in real canonical form.

To prove (7.1.2) in general we have to modify the inner product and norm. Here we will only prove it in the semisimple case. Then there is a basis $\mathcal{B} = \{\hat{e}_1, ..., \hat{e}_n\}$ putting $A$ in the form $A = \text{diag}\{\lambda_1, ..., \lambda_p, A_1, ..., A_m\}$ discussed above. Associated with this basis is an inner product as follows. We write $x = \hat{x}_1\hat{e}_1 + ... + \hat{x}_n\hat{e}_n$ and $y = \hat{y}_1\hat{e}_1 + ... + \hat{y}_n\hat{e}_n$ and $\langle x, y\rangle_{\mathcal{B}} = \hat{x}_1\hat{y}_1 + ... + \hat{x}_n\hat{y}_n$ and $|x|^2_{\mathcal{B}} = \langle x, x\rangle_{\mathcal{B}}$. Then in this basis by the previous argument $\langle x, Ax\rangle = a_1\hat{x}_1^2 + .... + a_n\hat{x}_n^2$. Therefore we conclude that

$$\alpha|x|^2_{\mathcal{B}} \leq \langle x, Ax\rangle_{\mathcal{B}} \leq \beta|x|^2_{\mathcal{B}} \tag{7.1.7}$$

which by the same argument as before gives

$$e^{\beta t}|x(0)|_{\mathcal{B}} \leq |x(t)|_{\mathcal{B}} \leq e^{\beta t}|x(0)|_{\mathcal{B}} \tag{7.1.8}$$

To prove (7.1.2) we must show that the norms are equivalent, i.e. that there is a constant $C$ such that

$$C^{-1}|x| \leq |x|_{\mathcal{B}} \leq C|x| \tag{7.1.9}$$

Since $\hat{x} = Qx$ and

$$|x|_{\mathcal{B}} = |\hat{x}| = |Qx| \leq \|Q\|\,|x|, \qquad \text{where} \qquad \|Q\| = \sup_x \frac{|Qx|}{|x|} \leq C \tag{7.1.10}$$

Similarly $x = Q^{-1}\hat{x}$ so

$$|x| = |Q^{-1}\hat{x}| \leq \|Q^{-1}\|\,|x|, \qquad \text{where} \qquad \|Q^{-1}\| = \sup_x \frac{|Q^{-1}x|}{|x|} \leq C \tag{7.1.10}$$

for some possibly larger constant $C$.