Lecture 1: Overview. What are differential equations about? An ordinary differential equation is a relation

$$F(x''(t), x'(t), x(t), t) = 0 \tag{0.1}$$

determined by the physics of the problem. $x(t)$ could be the position x of a particle at time t and (0.1) could be Newton's law: $m\ddot{x} = F(x)$. (mass times acceleration is equal to the force) An example is the equation of a spring where $F(x) = -kx$. We want to say something about how solutions to (0.1) behave. We might want to know e.g. if the particle will escape to infinity or if it will stay in a bounded region.

Section 1.1:. The simplest example is

$$\frac{dx}{dt} = ax \tag{1.1}$$

where a is a constant. For any constant K

$$x(t) = Ke^{at} \tag{1.2}$$

is a solution to (1.1). In fact then $x'(t) = aKe^{at} = ax(t)$. Moreover, there are no other solutions. In fact if x satisfies (1.1) and we multiply with an integrating factor e^{-at} we get

$$\frac{d}{dt}\big(x(t)e^{-at}\big) = x'(t)e^{-at} - ax(t)e^{-at} = e^{-at}\big(x'(t) - ax(t)\big) = 0 \tag{1.3}$$

Therefore $x(t)e^{-at}$ is a constant K, so $x(t) = Ke^{at}$.

The constant K is completely determined if we also give the value of x_0 of the solution at some time t_0, so $x(t_0) = x_0$. In particular if we choose $t_0 = 0$ we get an initial value problem:

$$\frac{dx}{dt} = ax, \qquad x(0) = K \tag{1.4}$$

and the only solution is given by (1.2).

The constant a in (1.1) is to be considered as a parameter. If a changes, then the equation and hence the solution changes. The sign of a is crucial:

If $a > 0$, then $\lim_{t\to\infty} Ke^{at} = \pm\infty$ depending on if $K > 0$ or $K < 0$.

If $a = 0$, then $Ke^{at} = \text{constant}$.

If $a < 0$, then $\lim_{t\to\infty} Ke^{at} = 0$.

The qualitative behavior of solutions is illustrated by sketching the graphs of solutions.

The equation (1.1) is unstable when $a = 0$ in the sense that the slightest change in the parameter a gives rise to large change in the solutions behavior for large times. We will also say that $a = 0$ is a *bifurcation point* in the one-parameter family of equations (1.1) depending on the parameter a.