Lecture 20: 9.1 Nonlinear Sinks. Consider a nonlinear differential equation

$$(9.1.1) \qquad x' = f(x), \qquad \text{where} \quad f : W \to \mathbf{R}^n, \quad \text{and} \quad W \subset \mathbf{R}^n, \quad \text{is open}$$

We also assume that f is continuously differentiable.

A point $\overline{x} \in W$ is called an *equilibrium* point of (9.1.1) if $f(\overline{x}) = 0$. Then the constant function $x(t) = \overline{x}$ is a solution to (9.1.1). By uniqueness no other solution curve can pass through $\overline{x}$ (see Chapter 8 of book).

Let $\Omega \subset \mathbf{R} \times W$ be an open set. The flow associated with (9.1.1) $\phi : \Omega \to W$ is defined by that for each $x \in W$ the map $t \to \phi(t, x) = \phi_t(x)$ is a solution to (9.1.1) passing through x when $t = 0$. $\overline{x}$ such that $\phi_t(\overline{x}) = \overline{x}$ is also called *stationary point* or *fixed point* of the flow of the differential equation. Another name for $\overline{x}$ is a *zero* or singular point for the vector field f.

If $f(x) = Ax$ is linear then the origin $x = 0$ is an equilibrium. If all eigenvalues have negative real part $\leq \lambda < 0$ then ϕ_t approaches 0 exponentially; $|\phi_t(x)| \leq Ce^{\lambda t}$.

Suppose that $f \in C^1$ and $f(0) =$. Then we can think of the derivative $Df(0) = A$ of f at 0 as the linear vector field that best approximates f close to $x = 0$. We call it the *linear part* of f at 0. If all eigenvalues of f have negative real part we call 0 a *sink* More generally, an equilibrium $\overline{x}$ is a sink if all eigenvalues of $Df(\overline{x})$ have negative real part. The following theorem says that a nonlinear sink $\overline{x}$ behaves locally like a linear sink: nearby solutions approach $\overline{x}$ exponentially.

Theorem. *Let $\overline{x} \in W$ be a sink of (9.1.1). Suppose that every eigenvalue of $DF(\overline{x})$ has real part $< -c < 0$. Then there is a neighborhood $U \subset W$ of $\overline{x}$ such that $\phi_t(x)$ is defined and in U for all $x \in U$, $t > 0$ and there is a constant K such that*

$$(9.1.2) \qquad |\phi_t(x) - \overline{x}| \leq Ke^{-tc}|x - \overline{x}|, \qquad \textit{for all } x \in U, \ t \geq 0$$

Proof. We may assume that $\overline{x} = 0$. Set $A = Df(0)$. Choose $-b < -c < 0$ so all the eigenvalues of A have real part less than $-b$. By section 7.1 we find a basis $\mathcal{B}$ with associated inner product and norm such that

$$(9.1.3) \qquad \langle x, Ax \rangle_{\mathcal{B}} \leq -b|x|^2_{\mathcal{B}}$$

Since $A = Df(0)$ we have $\lim_{x \to 0} |f(x) - Ax|/|x| = 0$ and hence
$\lim_{x \to 0} |\langle f(x) - Ax, x|/|x|^2 = 0$ so for small $|x|$:

$$(9.1.3) \qquad \langle f(x), x \rangle_{\mathcal{B}} \leq -c|x|^2_{\mathcal{B}}$$

As in section 7.1:

$$(9.1.4) \qquad \frac{d}{dt}|x|_{\mathcal{B}} = \frac{1}{|x|_{\mathcal{B}}}\langle x', x \rangle_{\mathcal{B}} \leq -c|x|_{\mathcal{B}}$$

and hence as in section 7.1

$$(9.1.5) \qquad |x(t)|_{\mathcal{B}} \leq e^{-ct}|x(0)|_{\mathcal{B}}$$

By the equivalence of norms in section 7.1 this proves the theorem. $\square$