Lecture 21: 9.1 Nonlinear Sinks and The Pendulum. Let us consider the nonlinear system describing the motion of the pendulum. Let ℓ be the length of the rod and m be the mass of the bob. The bob moves along a circle of radius ℓ. If $\theta(t)$ is the counterclockwise angle from the vertical to the rod at time t, then the angular velocity of the rod is θ' and then the velocity of the bob is $\ell\,\theta'$. The force on the bob tangent to the circle of motion is

$$F = F(\theta, \theta') = -k\ell\theta' - mg\sin\theta \tag{9.1.6}$$

Here the first part is due to friction; $k \geq 0$ is a constant The second part is the tangential component of the gravitational force; $g > 0$ is the gravitational constant.

The tangential acceleration of the bob is $a = \ell\theta''$. Hence by Newton's law, $ma = F$:

$$\theta'' = -\frac{k}{m}\theta' - \frac{1}{\ell}\sin\theta \tag{9.1.7}$$

Introducing the angular velocity as a new variable

$$\omega = \theta' \tag{9.1.8}$$

the second order equation (9.1.7) can be written as a first order system:

$$\theta' = \omega \tag{9.1.9}$$

$$\omega' = -\frac{k}{m}\theta' - \frac{1}{\ell}\sin\theta \tag{9.1.10}$$

or written as a vector equation

$$(\theta', \omega') = f(\theta, \omega), \qquad \text{where} \quad f(\theta, \omega) = \Big(\omega, -\frac{k}{m}\theta' - \frac{1}{\ell}\sin\theta\Big) \tag{9.1.11}$$

This nonlinear system has equilibrium $f(\theta, \omega) = 0$, at the points $(\theta, \omega) = (n\pi, 0)$, $n = 0, \pm 1, \pm 2, ...$ The derivative of the vector field f is

$$Df(\theta, \omega) = \begin{bmatrix} 0 & 1 \\ -\cos\theta/\ell & -k/m \end{bmatrix} \tag{9.1.12}$$

Hence at the point $(0, 0)$ where the rob is hanging down vertically,

$$Df(0, 0) = \begin{bmatrix} 0 & 1 \\ -1/\ell & -k/m \end{bmatrix} \tag{9.1.13}$$

with eigenvalues

$$-\frac{k}{2m} \pm \Big(\Big(\frac{k}{2m}\Big)^2 - \frac{4}{\ell}\Big)^{1/2} \tag{9.1.14}$$

The real part is < 0 as long as $k > 0$ so $(0, 0)$ is a sink then. There we conclude that the fixed point $(0, 0)$ is stable, i.e. that all solutions starting sufficiently close to $(0, 0)$ will approach $(0, 0)$ as $t \to \infty$. Note that this is what we could have guessed by making the linear approximation $\sin\theta \sim \theta$.

Note also that the fixed point $(\pi, 0)$ corresponds to the rob standing up vertically. This one is unstable. In fact the derivative at that point is given by (9.1.13) with ℓ replaced by $-\ell$ and the eigenvalues are hence given by (9.1.14) with ℓ replaced by $-\ell$. Since the large eigenvalue

$$-\frac{k}{2m} + \Big(\Big(\frac{k}{2m}\Big)^2 + \frac{4}{\ell}\Big)^{1/2} > 0 \tag{9.1.15}$$

the corresponding linearized system is unstable.

Note also an apparent contradiction that the pendulum never comes to rest but continuous to move closer and closer to the equilibrium