Lecture 22: Section 9.2: Stability. The most natural notion of stability of solutions to

$$x' = f(x) \tag{9.2.1}$$

is that a equilibrium point is stable if all nearby solutions stay nearby for all future times. The precise mathematical definition is as follows. An equilibrium $\overline{x}$ is called *stable* if for every neighborhood U of $\overline{x}$ there is a neighborhood U_1 of $\overline{x}$ in U such that $x(0) \in U_1$ implies that (9.2.1) has a solution x with $x(t) \in U$ for all $t > 0$. If U_1 can be chosen so that in addition $\lim_{t\to\infty} x(t) = \overline{x}$ for $x(0) \in U_1$, the $\overline{x}$ is *asymptotically stable*. An equilibrium $\overline{x}$ that is not stable is called *unstable*. This mean that there is a neighborhood U of $\overline{x}$ such that for every neighborhood U_1 of $\overline{x}$ there is at least one solution starting at $x(0) \in U_1$ which does not lie entirely in U. (Note that in the pictures in the book U_1 and U have been switched around.)

A sink is asymptotically stable and therefore stable. An example of an equilibrium that is stable but not asymptotically stable is a linear system with imaginary eigenvalues or the nonlinear pendulum without friction. The relevance of this is limited since the slightest perturbation of the system will destroy its character. Even a small linear perturbation can make it into a sink or source since hyperbolicity is a generic property. An example of an unstable equilibrium is a linear system with a source, i.e. with an eigenvalue with a positive real part. To complement the theorem in section 1 we have:

Theorem. *Suppose that $\overline{x}$ is a stable equilibrium point of (9.2.1), where $f \in C^1$. Then no eigenvalue of $Df(\overline{x})$ has positive real part.*

The proof is a bit long and uses ideas from Chapter 8 that we have not covered. Since the rest of the book is independent of the proof we will skip it.

We will instead prove a weaker statement: If all eigenvalues of $Df(\overline{x})$ have positive real part $> a > 0$ then the system is unstable. Let us for simplicity assume that the equilibrium point is $\overline{x} = 0$. The proof of this is essentially the same as that of Theorem 9.1 noting that the opposite inequalities now hold

$$\langle f(x), x\rangle_{\mathcal{B}} > a|x|_{\mathcal{B}}^2, \qquad \text{if} \qquad |x| \le c_1 \tag{9.2.2}$$

and hence

$$\frac{d}{dt}|x|_{\mathcal{B}} \ge a|x|_{\mathcal{B}}, \qquad \text{if} \qquad |x| \le c_1. \tag{9.2.3}$$

Let us now assume that $\overline{x} = 0$ is stable, in which case there is a time t_1 such that $|x(t)| \le c_1/2$ for $t \ge t_1$. Then it follows from (9.2.3) that

$$|x(t)|_{\mathcal{B}} \ge e^{a(t-t_1)}|x(t_1)|_{\mathcal{B}}, \qquad \text{for} \quad t \ge t_1 \tag{9.2.4}$$

but since $a > 0$ this leads to a contradiction since (9.2.4) implies that $|x(t)| \to \infty$ as $t \to \infty$ since $x(t_1) \ne 0$.

Let us introduce one more concept. If $\overline{x}$ is an equilibrium point then the union of all solution curves $x(t)$ that tend to $\overline{x}$ as $t \to \infty$ is called the *basin* of $\overline{x}$ and is denoted by $B(\overline{x})$.