Lecture 26: 10.1-10.2 The electric circuit. We study the RLC circuit, of a resistor R, an inductor L and a capacitor C, in series. We first define directions in which the currencies through and voltage over each component are measured. Kirchoff's first law say that the total currencies into each node is equal to the total currencies out of each node

$$i_R = i_L = -i_C \tag{10.1.1}$$

and Kirchoff's second law says that the total voltage drop over a loop, i.e. the sum of the voltage drops over each component in a loop, is zero:

$$v_R + v_L - v_C = 0 \tag{10.1.2}$$

Furthermore, Faraday's law says that

$$L\frac{di_L}{dt} = v_L \tag{10.1.3}$$

and the capacitor satisfies

$$C\frac{dv_C}{dt} = i_C \tag{10.1.4}$$

and we assume that the resistor satisfies a generalized Ohm's law

$$f(i_R) = i_L \tag{10.1.5}$$

The differential equations are therefore the system:

$$L\frac{di_L}{dt} = v_L = v_c - f(i_L) \tag{10.1.6}$$

$$C\frac{dv_C}{dt} = i_C = -i_L \tag{10.1.7}$$

Section 10.2. If we let $x = i_L$ and $y = v_C$ we get the system

$$(x', y') = F(x, y) = \big(y - f(x), -x\big) \tag{10.2.1}$$

This is *Lienard's equation.* If $f(x) = x^3 - x$ it is called *Van der Pol's equation.*

The case of $f = Kx$, $K > 0$, is an ordinary resistor satisfying Ohm's law:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -K & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{10.2.2}$$

with eigenvalues $\lambda = \big(-K \pm (K^2-4)^{1/2}\big)/2$. Since λ has negative real part $(0,0)$ is an asymptotically stable equilibrium.

In general the critical points of (10.2.1) are $(0, f(0))$. The derivative matrix is

$$DF(0, f(0)) = \begin{bmatrix} -f'(0) & 1 \\ -1 & 0 \end{bmatrix} \tag{10.2.3}$$

with eigenvalues $\lambda = \big(-K \pm (K^2-4)^{1/2}\big)/2$, where $K = f'(0)$. Hence $(0, f(0))$ is a sink if $f'(0) > 0$ and a source if $f'(0) < 0$.

Proposition. *Let $W(x,y) = x^2 + y^2$. If $(x(t), y(t))$ be a solution to (10.2.1) then*

$$\frac{d}{dt}W(x(t), y(t)) = -x(t)f(x((t)) \tag{10.2.4}$$

Proof. By the chain rule $dW(x(t), y(t))/dt = x(t)x'(t) + y(t)y'(t)$ and substituting from (10.2.1) gives (10.2.4).

The interpretation of this is that energy decreases according to the power dissipated by the resistor. A resistor is called passive if $xf(x) < 0$ except when $x = 0$. In this case it follows that W is a Liapunov function and from Theorem 9.3.2 it follows that the origin is asymptotically stable and its basin is the whole plane.

Section 10.3: Van der Pol's Equation.

$$x' = y - f(x), \qquad f(x) = x^3 - x \tag{10.3.1}$$

$$y' = -x \tag{10.3.2}$$

In this section we will prove the following theorem:

Theorem. *There is one nontrivial periodic solution of (10.3.1)-(10.3.2) and every nonequilibrium solution tends to this periodic solution. "The system oscillates".*

We know from the previous section that the only equilibrium is $(0, f(0)) = (0, 0)$ and it is a source since $f'(0) = -1 < 0$. The next step is to show that every nonequilibrium solution "rotates" in a certain sense around the equilibrium in a clockwise direction. To prove this we divide the x-y plane into four regions

$$\begin{aligned} A &= \{(x, y); y > f(x),\, x > 0\}, \qquad & B &= \{(x, y); y < f(x),\, x > 0\}, \\ C &= \{(x, y); y < f(x),\, x < 0\}, \qquad & D &= \{(x, y); y > f(x),\, x < 0\}. \end{aligned} \tag{10.3.3}$$

with boundaries

$$\begin{aligned} v^+ &= \{(x, y); y > 0, x = 0\}, \qquad & v^- &= \{(x, y); y < 0, x = 0\}, \\ g^+ &= \{(x, y); x > 0, y = f(x)\}, \qquad & g^- &= \{(x, y); x < 0, y = f(x)\}. \end{aligned} \tag{10.3.4}$$

The vector (x', y') is pointing horizontally to the right on v^+, horizontally to the left on v^-, vertically up on g^-, vertically down on g^+. The signs of x' and y' are constant in each region A, B, C, D.

Proposition 1. *Any trajectory starting in v^+ enters A. Any trajectory starting in A meets g^+, furthermore it meets g^+ before it meets v^-, g^- or v^+.*

Proof. Let $(x(t), y(t))$ be a solution curve to (10.3.1)-(10.3.2). If $(x(0, y(0)) \in v^+$ then $x(0) = 0$ and $y(0) > 0$. Since $x'(0)$, $x(t)$ increases for small t so $x(t) > 0$ which implies that $y(t)$ decreases for small t. Hence the curve enters A. Before the curve leaves A (if it does), x' must become 0 again, so the curve must cross g^+ before it meets $v-$, g^- or v^+. Thus the first and last statements of the proposition are proved. It remains to show that a curve starting inside A; $(x(0), y(0)) \in A$, reaches g^+; $(x(t), y(t)) \in g^+$ for some positive time $t > 0$. If it doesn't then it must remain in A and furthermore $y(t) \leq y(0)$. (Since this is a bounded set it follows from Theorem 8.5.1 that the solution must exist for all times if it stays in this bounded region.) Since $x' > 0$ in A, $x(t) \geq x(0) = a > 0$, for $t > 0$. Hence from (10.3.2) it follows that $y'(t) \leq -a$, for $t > 0$. In that case $y(y) = \int_0^t y'(s)\, ds + y(0) \leq y(0) - at \to -\infty$, as $t \to \infty$. But this is impossible unless the trajectory meets g^+. $\square$

Similar arguments prove:

Proposition 2. *Every trajectory is defined for all $t \geq 0$. Except for $(0, 0)$, each trajectory repeatedly crosses the curves v^+, g^+, v^-, g^-, in a clockwise order, passing among the regions A, B, C, D in clockwise order.*