

Lecture 27: Section 10.3 Cont. To analyze the Van der Pol oscillator we define:

$$(10.3.5) \quad \sigma : v^+ \rightarrow v^+$$

as follows. For $p \in v^+$ let $\phi_t(p)$ be the solution curve starting at p and let $t_1 = t_1(p) > 0$ be the first time that the solution curve meets v^+ again. We put $\sigma(p) = \phi_{t_1}(p)$. One can prove that the map (10.3.5) is continuous. σ is called the *section map*

Proposition 3. *Let $p \in v^+$. Then p is a fixed point of σ (i.e. $\sigma(p) = p$) if and only if p is on a periodic solution of (10.3.1). (i.e. $\phi_t(p) = p$, for some $t \neq 0$) Moreover every periodic solution curve meets v^+ .*

Proof. If $\sigma(p) = p$ then the orbit is periodic and we will show that if $\sigma(p) \neq p$ then $\sigma^k(p) = \sigma \circ \dots \circ \sigma(p) \neq p$ (k -times), for any $k \geq 1$, i.e. the orbit is not periodic. If $\sigma(p) > p$ then $\sigma^2(p) > \sigma(p)$, because otherwise the trajectory from $\sigma(p)$ to $\sigma^2(p)$ would have to intersect the trajectory from p to $\sigma(p)$. The last statement follows from Proposition 2. \square

Similarly we define a map

$$(10.3.6) \quad \alpha : v^+ \rightarrow v^-$$

as follows. For $p \in v^+$ let ϕ_t be the trajectory starting at p and let $t_2 = t_2(p) > 0$ be the first time the curve meets v^- . We put $\alpha(p) = \phi_{t_2}(p)$. We claim that if $\alpha(p) = -p$ then $\sigma(p) = p$. In fact since the right hand side of (10.3.1)-(10.3.2) changes sign when (x, y) is replaced by $(-x, -y)$ it follows that if $t \rightarrow (x(t), y(t))$ is a solution then $t \rightarrow (-x(t), -y(t))$ is also a solution, so the trajectory $0 \leq t \leq t_1$ determines the trajectory $t_1 \leq t \leq t_2 = 2t_1$. Let

$$(10.3.7) \quad \delta(p) = (|\alpha(p)|^2 - |p|^2)/2$$

Let $r = |p_0|$, where p_0 is the number such that the trajectory starting at p_0 first hits the x -axis at $(1, 0)$. it follows from the following propositions that $\delta(p) = 0$ for some p which proves the existence of a periodic orbit.

Proposition 4. (a) $\delta(p) > 0$, if $0 < |p| < r$,
 (b) $\delta(p)$ decreases monotonously to $-\infty$, as $|p| \rightarrow \infty$, $|p| \geq r$.

Proof. Let $W(x, y) = (x^2 + y^2)/2$. Let $(x(t), y(t))$ be the solution curve from $(p, 0) = (x(0), y(0))$ to $(\alpha(p), 0) = (x(t_2), y(t_2))$. Then by the Proposition 10.2.1

$$(10.3.8) \quad \delta(p) = \int_0^{t_2} \frac{d}{dt} W(x, y) dt = \int_0^{t_2} -x(x^3 - x) dt = \int_0^{t_2} x^2(1 - x^2) dt$$

In $|p| < r$ then $0 < x < 1$ and so the integrand is positive which proves (a). If $|p| \geq r$ we define $y_2 > y_1$ to be the y coordinate so $(1, y_i) = (x(s_i), y(s_i))$, $i = 1, 2$ is on the curve at time s_i . Then

$$(10.3.9) \quad \int_0^{s_1} x^2(1 - x^2) dt = \int_0^1 \frac{x^2(1 - x^2)}{dx/dt} dx = \int_0^1 \frac{x^2(1 - x^2)}{y - f(x)} dx$$

is a decreasing function of p since increasing p means increasing $y = y(x)$ for fixed x . Similarly for the integral from s_2 to t_2 . We have

$$(10.3.10) \quad \int_{s_1}^{s_2} x^2(1 - x^2) dt = \int_{y_1}^{y_2} \frac{x^2(1 - x^2)}{dy/dt} dy = \int_{y_1}^{y_2} x(y)(1 - x(y)^2) dy$$

As p increases the domain $[y_1, y_2]$ increases and $x(y) = x_p(y)$ increases as p increases since the curve moves to the right. Hence (10.3.10) decreases with increasing p . \square

Section 10.4: Hopf Bifurcation. In this section we consider differential equations depending on a *parameter*, μ . We are interested in how the phase portrait changes as μ varies. A value μ_0 where there is a basic structural change in the phase portrait is called a *bifurcation* point. We consider:

$$(10.4.1) \quad x' = y - f_\mu(x), \quad f_\mu(x) = x^3 - \mu x$$

$$(10.4.2) \quad y' = -x$$

and see what happens when μ varies from -1 to 1 . For $1 \leq \mu \leq 0$ the resistor is passive and the proposition of Section 2 implies that all solutions tend asymptotically to zero as $t \rightarrow \infty$; the origin is a sink at least when $-1 \leq \mu < 0$. However, when $0 < \mu \leq 1$ the resistor is active and it follows from the analysis of section 3 that there will be a unique periodic solution γ_μ and the origin becomes a source. In fact every solution, apart from the unstable equilibrium solution $(0, 0)$, tend to γ_μ as $t \rightarrow \infty$. One can also show that $\gamma_\mu \rightarrow 0$ as $\mu \searrow 0$. The bifurcation value is $\mu = 0$. The basic structure of the system changes as μ passes through the value 0 . E. Hopf proved that for fairly general one-parameter families of equations $x' = f_\mu(x)$, there must be a closed orbit for $\mu > \mu_0$ if the eigenvalues character of an equilibrium changes suddenly at μ_0 from a sink to a source.