Lecture 2: 1.1 Cont.. Consider next a system of two equations

$$x_1' = a_1 x_1, \qquad x_2' = a_2 x_2 \tag{1.5}$$

which has the solutions

$$x_1(t) = K_1 e^{a_1 t}, \qquad x_2(t) = K_2 e^{a_2 t} \tag{1.6}$$

From a geometric point of view we think of $x(t) = (x_1(t), x_2(t))$ as a curve in the plane $\mathbf{R}^2$. Using vector notation we can write (1.5) as

$$x' = Ax \tag{1.7}$$

where $x'(t) = ((x_1'(t), x_2'(t))$ is the *tangent vector* to the curve and Ax is the vector $(a_1 x_1, a_2 x_2)$. Initial conditions is hence also a vector $x(t_0) = u$. The right hand side of (1.7) can hence be thought of as a *vector field*, i.e. for each point $x = (x_1, x_2)$ in $\mathbf{R}^2$ we assign a vector Ax. One can illustrate the vector field graphically by for a few values of x draw the vector from x to $x + Ax$.

Solving the differential equation (1.5) with initial conditions $u = (u_1, u_2)$ at $t = 0$ means finding a curve $x(t)$ that satisfies (1.5) passing through the point u when $t = 0$. Graphically one can draw the curves for a few starting points. The family of solution curves as subsets of $\mathbf{R}^2$ is called the "phase portrait" of the equation. The solution curves to (1.7) can be sketch graphically from the sketch of the vector field $x \to Ax$. In fact (1.7) says that the tangent line to the curve $x(t)$ should point in the direction of the vector field $x \to Ax$ at the point $x(t)$. Therefore one can sketch the curve by at each point along the curve following the direction of the vector field.

The graph of the solution for (1.5) requires a picture in 3 dimensions so it is easier to sketch the phase portrait.

Ex. For $a_1 = 1$ and $a_2 = -1/2$ sketch the vector field $Ax = (a_1 x_1, a_2 x_2)$ and the phase portrait for the solutions of (1.5).

We can think of the equations (1.5) as a *dynamical system*. This means that t is interpreted as time and the curve $x(t)$ is to be thought of as the path of a particle. The solution of (1.7) at time t with initial condition $x(0) = u$ is given by $x(t) = \phi_t(u)$ where

$$\phi_t(u) = (u_1 e^{a_1 t}, u_2 e^{a_2 t}) \tag{1.8}$$

We can imagine lots of dust particles moving. The particle at location u at time 0 will at time t be located at $\phi_t(u)$. For fixed t, the transformation $u \to \phi_t(u)$ is linear and the collection of maps ϕ_t is a one-parameter family of transformations (depending on the parameter t). This is called the flow or dynamical system determined by the vector field $x \to Ax$.

Let us now study the more complicated system:

$$x_1' = 5x_1 + 3x_2, \qquad x_2' = -6x_1 - 4x_2 \tag{1.9}$$

We want to find a change of coordinates that transform it to the uncoupled or diagonal form of the previous example (1.5). We claim that in this case the map

$$y_1 = 2x_1 + x_2, \qquad y_2 = x_1 + x_2 \tag{1.10}$$

does the job. (In Chapter 3 it will be explained how one comes up with this change of coordinates.) Solving for x in terms of y

$$x_1 = y_1 - y_2, \qquad x_2 = -y_1 + 2y_2 \tag{1.11}$$

Differentiating (1.10) and substituting from (1.9) gives

$$y_1' = 2x_1' + x_2' = ... = 4x_1 + 2x_2, \qquad y_2' = x_1' + x_2\prime = ... = -x_1 - x_2 \tag{1.12}$$

If we also substitute from (1.11)

$$y_1' = 2y_1, \qquad y_2' = -y_2. \tag{1.13}$$

This system with initial conditions $(y_1(0), y_2(0)) = (v_1, v_2)$ has the solution

$$y_1(t) = v_1 e^{2t}, \qquad y_2(t) = v_2 e^{-t}. \tag{1.14}$$

The general solution to (1.9) is now given by substituting (1.14) into (1.11):

$$x_1(t) = v_1 e^{2t} - v_2 e^{-t}, \qquad x_2(t) = -v_1 e^{2t} + 2v_2 e^{-t} \tag{1.15}$$

Furthermore, the solution to (1.9) with initial conditions $(x_1(0), x_2(0)) = (u_1, u_2)$ is given by solving (1.10) for $(v_1, v_2) = (y_1(0), y_2(0))$ in terms of (u_1, u_2). Substituting $v_1 = 2u_1 + u_2$ and $v_2 = u_1 + u_2$ into (1.15) gives

$$x_1(t) = (2u_1 + u_2)e^{2t} - (u_1 + u_2)e^{-t}, \qquad x_2(t) = -(2u_1 + u_2)e^{2t} + 2(u_1 + u_2)e^{-t} \tag{1.16}$$

Section 1.2:. Consider the linear system

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n \end{aligned} \tag{1.17}$$

of n differential equations in n unknowns. This system can also be written as

$$x' = Ax \tag{1.18}$$

where

$$x'(t) = (x_1'(t), ..., x_n'(t)) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} \tag{1.19}$$

is the tangent to the curve $x(t)$ and the vector function $x \to Ax$ is given by matrix multiplication with the matrix

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ & \vdots & & \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}, \tag{1.20}$$

i.e. Ax is the vector whose ith coordinate is

$$a_{i1}x_1 + ... + a_{in}x_n \tag{1.21}$$