Lecture 3: Chapter 2:. If a particle of mass m is moving in force field F then the acceleration a of the particle is given by Newtown's Second Law, i.e

$$ma = F \tag{0.1}$$

If $x(t)$ is the position of the particle at time t then the acceleration is $\ddot{x}(t)$ and it is reasonable to assume that the force depends on the position x of the particle. Then we get a second order differential equation: $m\ddot{x} = F(x)$, or

$$\ddot{x} = \frac{1}{m}F(x) \tag{0.2}$$

Section 1: The Harmonic Oscillator. In the case of a a mass hanging in a spring and x denoting the displacement in height from the equilibrium position, the force acting on the mass is $F(x) = -kx$ where $k \geq 0$ is the spring constant. Hence

$$\ddot{x} + p^2 x = 0 \tag{1.1}$$

where $p = \sqrt{k/m}$. This has the solution

$$x = A\cos(pt) + B\sin(pt) \tag{1.2}$$

Another example is the pendulum. Let ℓ be the length of the rod and m be the mass of the bob. The bob moves along a circle of radius ℓ. If $\theta(t)$ is the angle from the vertical to the rod at time t, then the force on the bob tangent to the circle is $F(\theta) = -mg\sin\theta$ and the tangential acceleration is $a = \ell\ddot{\theta}$. Hence by (0.1)

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0 \tag{1.3}$$

For small θ we can use the approximation $\sin\theta \sim \theta$ to estimate the solution of (1.3) by the solution of (1.1), given by (1.2). The equation (1.1) is linear whereas (1.3) is nonlinear. For linear equations we will be able to find the solution in explicit form but for nonlinear equations this in general will not be the case. Instead for nonlinear equations we will be able to prove some qualitative properties like stability.

We also remark that there is a two dimensional version of the harmonic oscillator where $x(t) = (x_1(t), x_2(t))$ is a path in the plane and the force is the vector field $F(x) = -mkx$, where $x = (x_1, x_2)$. Newton's law gives the differential equation

$$\ddot{x} + k^2 x = 0 \tag{1.4}$$

Since the equations for the two components separate each of them taking the form (1.1), each component will be of the form (1.2) with different constants.

Section 2: Some Calculus. The inner product of $x, y \in \mathbf{R}^n$ is defined to be

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \tag{2.1}$$

The length of a vector is given by $|x|^2 = \langle x, x\rangle$. By Leibniz's rule:

$$\frac{d}{dt}\langle x, y\rangle = \langle x', y\rangle + \langle x, y'\rangle \tag{2.2}$$

The *gradient* of a function $f : \mathbf{R}^n \to \mathbf{R}$ is the vector

$$\operatorname{grad} f(x) = \Big(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \cdots, \frac{\partial f}{\partial x_n}(x)\Big) \tag{2.3}$$

It follows from the Chain rule that if $f : \mathbf{R} \to \mathbf{R}^n$ and $g : \mathbf{R}^n \to \mathbf{R}$ then

$$\frac{d}{dt}g\big(f(t)\big) = \big\langle \operatorname{grad} g\big(f(t)\big),\, f'(t)\,\big\rangle \tag{2.4}$$

Section 3: Conservative Force Fields. A vector field is called conservative if

$$F(x) = -\text{grad}\, V(x) \tag{3.1}$$

for some function V called a potential. The vector fields discussed in section 1 are all conservative, e.g. for the planar harmonic oscillator with force field $F(x) = -mkx$ the potential energy is $V(x) = mk|x|^2/2$.

For a moving particle of mass m the kinetic energy is defined to be

$$T = m|\dot{x}(t)|^2/2 \tag{3.2}$$

where $\dot{x}(t)$ is the velocity vector at time t and its length is the speed. For a particle moving in a conservative force field the potential energy is defined to be $V(x)$. The total energy or simply the energy is $E = T+V$. If $x(t)$ is the trajectory of a moving particle then the function

$$E(t) = \frac{1}{2}m|\dot{x}(t)|^2 + V(x(t)) \tag{3.3}$$

is in fact constant:

Theorem (Conservation of Energy). *Let $x(t)$ be the trajectory of a particle moving in a conservative force field $F(x) = -\text{grad}V(x)$, i.e. $x(t)$ satisfies $m\ddot{x} = F(x)$. Then the total energy (3.3) is independent of time.*

Proof. We need to show that the time derivative of (3.3) vanishes. First, by (2.2)

$$\frac{d}{dt}|\dot{x}|^2 = 2\langle \dot{x}, \ddot{x}\rangle \tag{3.4}$$

and by (2.4)

$$\frac{d}{dt}V(x) = \big\langle \text{grad } V(x),\, \dot{x} \,\big\rangle = -\langle F(x), \dot{x}\rangle \tag{3.5}$$

Hence

$$\frac{d}{dt}\Big(\frac{1}{2}m|\dot{x}|^2 + V(x)\Big) = \langle \dot{x}, m\ddot{x} - F(x)\rangle = 0 \quad \square \tag{3.6}$$

We remark that apart from that this applies to the two dimensional harmonic oscillator it also applies for a one dimensional system, when the vector x is a just in **R**. In particular for the pendulum we conclude that

$$E(t) = \dot{\theta}(t)^2 - \cos\theta(t) = E_0 \tag{3.7}$$

is constant. (One can also see this by direct differentiation.) It follows that

$$\frac{d\theta}{dt} = \pm\sqrt{E_0 + \cos\theta} \tag{3.8}$$

so

$$\frac{d\theta}{\sqrt{E_0 + \cos\theta}} = \pm dt \tag{3.9}$$

In principle the left hand side has an anti derivative $G(\theta)$, $G'(\theta) = 1/\sqrt{E_0 + \cos\theta}$ and the solution $\theta = \theta(t)$ is given by implicitly solving for θ in $G(\theta) = \pm t + c$.