

Lecture 5: C. Changes of bases and coordinates. Recall that $\{e_1, \dots, e_m\}$ is a basis for a vector space E if every vector $z \in E$ can be written in a unique way as a linear combinations of the basis vectors, i.e. there is $(x_1, \dots, x_n) \in \mathbf{R}^n$ such that

$$(3.1.16) \quad z = \sum_{i=1}^n x_i e_i,$$

and there is only one such (x_1, \dots, x_n) . For each $z \in E$ we hence get a unique $x \in \mathbf{R}^n$, we can therefore consider the coordinates x_i as functions of z :

$$(3.1.17) \quad \varphi : E \rightarrow \mathbf{R}^n, \quad \varphi(z) = (x_1(z), \dots, x_n(z)).$$

This the unique linear map that sends each $e_i \in E$ into the corresponding standard basis vector of \mathbf{R}^n . It follows that φ is an isomorphism, i.e. one-to-one and onto. Conversely, let $\varphi : E \rightarrow \mathbf{R}^n$ be an isomorphism and \bar{e}_i be the standard basis of \mathbf{R}^n . Then $e_i = \varphi^{-1}(\bar{e}_i)$, $i = 1, \dots, n$ is a basis of E such that $\varphi(\sum_{i=1}^n x_i e_i) = (x_1, \dots, x_n)$. We define a *coordinate system* on a vector space E to be an isomorphism $\varphi : E \rightarrow \mathbf{R}^n$. The coordinates of $z \in E$ are (x_1, \dots, x_n) , where $\phi(z) = (x_1, \dots, x_n)$. The coordinate functions x_i are the unique linear functions $E \rightarrow \mathbf{R}^n$ such that

$$(3.1.18) \quad x_i(e_j) = \delta_{ij}, \quad \text{where } \delta_{ij} = 1, \text{ if } i \neq j \text{ and } 1 \text{ if } i = j.$$

Let $\{e_1, \dots, e_n\}$ be a basis of E , (x_1, \dots, x_n) the corresponding coordinates and $\varphi : E \rightarrow \mathbf{R}^n$ the corresponding isomorphism. Let $\{\hat{e}_1, \dots, \hat{e}_n\}$ be another basis of E , $(\hat{x}_1, \dots, \hat{x}_n)$ the corresponding coordinates and $\hat{\varphi} : E \rightarrow \mathbf{R}^n$ the corresponding isomorphism. Each \hat{e}_i can be expressed as a linear combinations of the basis $\{e_1, \dots, e_n\}$ so

$$(3.1.19) \quad \hat{e}_i = \sum_{j=1}^n p_{ij} e_j, \quad P = [p_{ij}]$$

Each coordinate \hat{x}_i can be expressed as a linear function of the old coordinates x_ℓ :

$$(3.1.20) \quad \hat{x}_k = \sum_{\ell=1}^n q_{k\ell} x_\ell, \quad Q = [q_{ij}]$$

Q is the matrix of the linear operator $\hat{\varphi}\varphi^{-1} : \mathbf{R}^n \rightarrow \mathbf{R}^n$. How are P and Q related?

$$(3.1.21) \quad x_\ell(e_j) = \delta_{\ell j}, \quad \hat{x}_k(\hat{e}_i) = \delta_{ki}$$

Substituting (3.1.20) and (3.1.19) into the second equation in (3.1.21) gives

$$(3.1.22) \quad \delta_{ki} = \sum_{\ell=1}^n q_{k\ell} x_\ell \left(\sum_{j=1}^n p_{ij} e_j \right) = \sum_{\ell=1}^n \sum_{j=1}^n q_{k\ell} p_{ij} x_\ell(e_j) = \sum_{\ell=1}^n \sum_{j=1}^n q_{k\ell} p_{ij} \delta_{\ell j},$$

since x_ℓ is linear and satisfies the first equation in (3.1.21). Since $\delta_{\ell j} = 0$ if $j \neq \ell$:

$$(3.1.23) \quad \delta_{ki} = \sum_{j=1}^n q_{kj} p_{ij}$$

The transpose of a matrix $P = [p_{ij}]$ is defined to be matrix

$$(3.1.24) \quad P^t = [p_{ij}^t], \quad p_{ij}^t = p_{ji}$$

Hence (3.1.23) can be written

$$(3.1.25) \quad \delta_{ki} = \sum_{j=1}^n q_{kj} p_{ji}^t$$

which in matrix language says that $I = QP^t$, i.e we have proved:

$$(3.1.26) \quad Q = (P^t)^{-1} = (P^{-1})^t, \quad P = (Q^t)^{-1} = (Q^{-1})^t$$

D. Operators, bases and matrices. Recall that to each linear operator T on \mathbf{R}^n we associated a matrix $A = [a_{ij}]$ by

$$(3.1.27) \quad Te_j = \sum_{i=1}^n a_{ij}e_i, \quad \Leftrightarrow \quad a_{ij} = (Te_j)_i$$

where $\{e_1, \dots, e_n\}$ is the standard basis of \mathbf{R}^n and $(Te_j)_i$ denotes the i th coordinate. Equivalently the i th coordinate of Tx is

$$(3.1.28) \quad (Tx)_i = \sum_{j=1}^n a_{ij}x_j, \quad x = x_1e_1 + \dots + x_n e_n$$

Suppose now that $T: E \rightarrow E$ is an operator on a vector space E and $\{e_1, \dots, e_n\}$ is any basis. If a_{ij} are defined by (3.1.27) then the coordinates for Tx are computed by (3.1.28). Suppose that another system of coordinates $(\hat{x}_1, \dots, \hat{x}_n)$ is introduced in E , corresponding to a basis $\{\hat{e}_1, \dots, \hat{e}_n\}$. Let B be the matrix of T in these coordinates. How is B related to A ? If $z \in E$ is any point the two sets of coordinates are related

$$(3.1.29) \quad \hat{x} = Qx, \quad x = Q^{-1}\hat{x}$$

The image Tz also has two sets of coordinates Ax and $B\hat{x}$. Therefore

$$(3.1.30) \quad B\hat{x} = QAx.$$

Hence

$$(3.1.31) \quad B\hat{x} = QAQ^{-1}\hat{x}$$

for all $\hat{x} \in \mathbf{R}^n$, i.e.

$$(3.1.32) \quad B = QAQ^{-1}$$

Two $n \times n$ matrices A and B such that (3.1.32) hold for some Q are called *similar*.

E. Determinants, trace and rank. The *determinant* is the unique function $\text{Det}: M_n \rightarrow \mathbf{R}$, such that for any matrices A and B :

$$(3.1.33) \quad \text{Det}(AB) = (\text{Det } A)(\text{Det } B), \quad \text{Det } I = 1, \quad \text{Det } A \neq 0 \Leftrightarrow A \text{ is invertible}$$

The *trace* of a matrix $A = [a_{ij}]$ is defined to be the sum of the diagonal elements

$$(3.1.34) \quad \text{Tr } A = \sum_{i=1}^n a_{ii}$$

The *rank* of a matrix A is defined to be dimension of the *image* of A

$$(3.1.35) \quad \text{rank } A = \dim(\text{Im } A)$$

All these numbers are invariant under similarity transformations:

$$(3.1.36) \quad \text{Det}(RAR^{-1}) = \text{Det } A, \quad \text{Tr}(RAR^{-1}) = \text{Tr } A, \quad \text{rank}(RAR^{-1}) = \text{rank } A$$

F. Direct sum decomposition. Let E_1, \dots, E_r be subspaces of E . We say that E is a direct sum of E_1, \dots, E_n if every x in a unique way can be written

$$(3.1.37) \quad x = x_1 + \dots + x_r, \quad x_i \in E_i$$

This is denoted by

$$(3.1.38) \quad E = E_1 \otimes \dots \otimes E_r$$

Let $T: E \rightarrow E$ be a linear operator and suppose that E_i is invariant under T , i.e. $TE_i \subset E_i$. Then we write $T = T_1 \otimes \dots \otimes T_r$ where $T_i = T|_{E_i}$. It follows that

$$(3.1.39) \quad \text{Det}(T_1 \otimes \dots \otimes T_r) = (\text{Det } T_1) \dots (\text{Det } T_r), \quad \text{Tr}(T_1 \otimes \dots \otimes T_r) = \text{Tr}(T_1) + \dots + \text{Tr}(T_r).$$