Lecture 6: Section 3.2 Diagonalizing a matrix with real eigenvalues. Our aim is to find a basis or system of coordinates in which a given operator T takes a simple diagonal form. A nonzero vector $z \in E$ is called an *eigenvector* and λ a real *eigenvalue* for T if $Tz = \lambda z$, for some real number λ. The hope is that we will be able to find a basis of eigenvectors for T in which case the matrix for T in this basis takes the simple diagonal form. That λ is an eigenvalue means that the kernel of

$$T - \lambda I : E \to E, \tag{3.2.1}$$

called the λ-*eigenspace*, is nontrivial, i.e. contains a nonvanishing vector. That the kernel of (3.2.1) is nontrivial is by the previous section equivalent to that

$$\text{Det}(T - \lambda I) = 0 \tag{3.2.2}$$

To find all eigenvalues we therefore pick a matrix A representing T and define

$$p(\lambda) = p_A(\lambda) = \text{Det}(A - \lambda I) \tag{3.2.3}$$

By the rules for calculating determinants in terms of lower order determinants by expanding along a row, this is a polynomial of degree $n = \dim E$, which we call the *characteristic polynomial* of A. It is invariant under similarity transformations of A:
(3.2.4)

$$p_{QAQ^{-1}}(\lambda) = \text{Det}(QAQ^{-1} - \lambda I) = \text{Det}\left(Q(A - \lambda I)Q^{-1}\right) = \text{Det}(A - \lambda I) = p_A(\lambda)$$

Hence it independent of changes of basis so it is independent of the particular representative of T and we can call it the characteristic polynomial of T. The real eigenvalues of T are exactly the real roots of $p(\lambda)$. A complex root of $p(\lambda)$ is called a complex eigenvalue of T. Once a real eigenvalue has been found:

$$p_A(\lambda) = 0 \tag{3.2.5}$$

we find the corresponding eigenvectors by solving

$$(A - \lambda)x = 0. \tag{3.2.6}$$

Ex. Find the eigenvalues and eigenvectors for the operator T which in the standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$ in $\mathbf{R}^2$ has the matrix considered in Chapter 1:

$$A = \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix}, \qquad \text{i.e.} \qquad Te_1 = 5e_1 - 6e_2, \quad Te_2 = 3e_1 - 4e_2 \tag{3.2.7}$$

Note if we write $x = x_1 e_1 + x_2 e_2$ and $Tx = (Tx)_1 e_1 + (Tx)_2 e_2$ then (3.2.7) says

$$\begin{bmatrix} (Tx)_1 \\ (Tx)_2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3.2.8}$$

Express T in the new basis and give the matrix for the change of coordinates.
Sol. The characteristic polynomial is

$$\text{Det} \begin{bmatrix} 5 - \lambda & 3 \\ -6 & -4 - \lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) \tag{3.2.9}$$

so the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$. The eigenvector(s) belonging to the eigenvalue $\lambda_1 = 2$ are solutions to the equation $(A - 2I)x = 0$, or

$$(3.2.10) \qquad \begin{bmatrix} 3 & 3 \\ -6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \Leftrightarrow \qquad \begin{cases} 3x_1 + 3x_2 = 0 \\ -6x_1 - 6x_2 = 0 \end{cases}$$

The solutions are $x_1 = t$ and $x_2 = -t$, $t \in \mathbf{R}$. Thus $\hat{e}_1 = (1, -1) \in \mathbf{R}^2$ is a solution for the eigenspace with eigenvalue 2. The eigenvector(s) belonging to the eigenvalue $\lambda_1 = 2$ are solutions to the equation $(A + I)x = 0$, or

$$(3.2.11) \qquad \begin{bmatrix} 6 & 3 \\ -6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \Leftrightarrow \qquad \begin{cases} 3x_1 + 3x_2 = 0 \\ -6x_1 - 6x_2 = 0 \end{cases}$$

The solutions are $x_1 = -t$ and $x_2 = 2t$, $t \in \mathbf{R}$ Thus $\hat{e}_2 = (-1, 2) \in \mathbf{R}^2$ is a solution for the eigenspace with eigenvalue -1. The two eigenvectors $\hat{e}_1$ and $\hat{e}_2$ are linearly independent so they form a basis for $\mathbf{R}^2$. In this basis T has the diagonal form

$$(3.2.12) \qquad \hat{A} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{i.e.} \qquad T\hat{e}_1 = 2\hat{e}_1, \quad T\hat{e}_2 = -\hat{e}_2,$$

from which it follows that $T(\hat{x}_1\hat{e}_1 + \hat{x}_2\hat{e}_2) = -\hat{x}_1\hat{e}_1 + 2\hat{x}_2\hat{e}_2$. The basis $\hat{e}_1$, $\hat{e}_2$ can be expressed in terms of the standard basis $e_1 = (1, 0)$, $e_2 = (0, 1)$:

$$(3.2.13) \qquad \hat{e}_1 = e_1 - e_2, \qquad \hat{e}_1 = -e_1 + 2e_2, \qquad \text{i.e} \qquad P = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

is matrix for the change of basis. What is the matrix for the change of coordinates:

$$(3.2.14) \qquad x_1e_1 + x_2e_2 = \hat{x}_1\hat{e}_1 + \hat{x}_2\hat{e}_2$$

The way to get hold of it is to substitute (3.2.13) into (3.2.14):

$$(3.2.15) \qquad x_1e_1 + x_2e_2 = (\hat{x}_1 - \hat{x}_2)e_1 + (-\hat{x}_1 + 2\hat{x}_2)e_2$$

i.e. $x_1 = \hat{x}_1 - \hat{x}_2$ and $x_2 = -\hat{x}_1 + 2\hat{x}_2$:

$$(3.2.16) \qquad x = S\hat{x}, \quad S = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \ (= P^t), \quad \hat{x} = S^{-1}x, \quad S^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

If we can find a basis of eigenvectors for T, $\{\hat{e}_1, ..., \hat{e}_n\}$ then the matrix for T in this basis is just the *diagonal* matrix $D = [d_{ij}] = \text{diag}\{\lambda_1, ..., \lambda_n\}$, i.e. $d_{ij} = 0$ if $i \neq j$ and $d_{ii} = \lambda_i$, the eigenvalue corresponding to $\hat{e}_i$. Then we say that T is *diagonalizable.*

Theorem 1. *Let T be an operator for an n-dimensional space E. If the characteristic polynomial of T has n distinct real roots, then T is diagonalizable.*

Proof. Let $\hat{e}_1, ..., \hat{e}_n$ be the eigenvectors with distinct eigenvalues $\lambda_1, ..., \lambda_n$. If they don't form a basis order them so $\hat{e}_1, ..., \hat{e}_m$ is a maximal independent set, $m < n$. Then $\hat{e}_n = \sum_{i=1}^{m} \hat{e}_i$ and
(3.2.17)

$$0 = (T - \lambda_n I)\hat{e}_n = \sum_{j=1}^{m} t_j(T - \lambda_n I)\hat{e}_j = \sum_{j=1}^{m} t_j(T\hat{e}_j - \lambda_n\hat{e}_j) = \sum_{j=1}^{m} t_j(\lambda_j - \lambda_n)\hat{e}_j$$

Since $\hat{e}_1, ..., \hat{e}_m$ are independent it follows that $t_j(\lambda_j - \lambda_n) = 0$, $j = 1, ..., m$. Since $\lambda_j \neq \lambda_n$ it follows that $t_j = 0$ so $\hat{e}_n = 0$ which is a contradiction. $\square$

Theorem 2. *Let A be an $n \times n$ matrix having n distinct real eigenvalues $\lambda_1, ..., \lambda_n$. Then there exists an invertible $n \times n$ matrix Q such that $QAQ^{-1} = \text{diag}\{\lambda_1, ..., \lambda_n\}$.*

Ex. The operators $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ have double eigenvalues 1. The first can be diagonalized but the second can not.