Lecture 7: Section 3.3: Differential equations with real, distinct eigenvalues. We will prove the following important result:

Theorem 3.1. *Let A be an operator on $\mathbf{R}^n$ having n distinct, real eigenvalues. Then for all $x_0 \in \mathbf{R}^n$, the linear equation*

$$x' = Ax, \qquad x(0) = x_0, \tag{3.3.1}$$

has a unique solution.

Proof. By Theorem 3.2.2 the conditions implies existence of an invertible matrix Q such that the matrix QAQ^{-1} is diagonal:

$$\hat{A} = QAQ^{-1} = \text{diag}\{\lambda_1, ..., \lambda_n\} \tag{3.3.2}$$

where $\lambda_1 < ... < \lambda_n$ are the eigenvalues of A. Introducing the new coordinates $\hat{x} = Qx$, $x = Q^{-1}\hat{x}$ we find

$$\hat{x}' = Qx' = QAx = QAQ^{-1}\hat{x} = \hat{A}\hat{x} \tag{3.3.3}$$

so

$$\hat{x}' = \hat{A}\hat{x} \tag{3.3.4}$$

Since $\hat{A}$ is diagonal this mean that

$$\hat{x}_i' = \lambda_i \, \hat{x}_i, \qquad i = 1, ..., n \tag{3.3.5}$$

Thus the equations are now uncoupled and we can solve each:

$$\hat{x}_i(t) = \hat{x}_i(0)e^{t\lambda_i} \tag{3.3.6}$$

To solve (3.3.1) put $\hat{x}(0) = Qx_0$ and

$$x(t) = Q^{-1}\hat{x}(t) \tag{3.3.7}$$

and

$$\hat{x}(t) = Q^{-1}\big(\hat{x}_1(0)e^{\lambda_1 t}, ..., \hat{x}_n(0)e^{\lambda_n t}\big) \tag{3.3.8}$$

Differentiation shows that

$$x' = Q^{-1}\hat{x}' = Q^{-1}\hat{A}\hat{x} = Q^{-1}(QAQ^{-1})\hat{x} = Ax \tag{3.3.9}$$

Moreover

$$x(0) = Q^{-1}\hat{x}(0) = Q^{-1}Qx_0 = x_0 \tag{3.3.10}$$

which shows that it solves (3.3.1). To prove the uniqueness we assume that we have two different solutions to (3.3.1) $u(t)$ and $v(t)$ with the same initial data x_0. Then the difference $w(t) = u(t) - v(t)$ is also a solutions to (3.3.1) but with vanishing initial data $w(0) = 0$. If we make the coordinate transformation $\hat{w} = Qw$ we get a solution to (3.3.4) with vanishing initial data. However the unique solutions to this equation are given by (3.3.6). This was proved already in Chapter 1 by using the integrating factor. This proves the theorem. □

It is important that the proof is constructive; we actually get the matrix Q. Let us review how we obtained it. Let us assume that we found n distinct, real eigenvalues and their eigenvectors

$$(A-\lambda_i I)\hat{e}_i = 0 \tag{3.3.11}$$

Then we write $\hat{e}_i$ in coordinates

$$\hat{e}_i = (p_{i1}, ..., p_{in}) = p_{i1}e_1 + ... + p_{in}e_n \tag{3.3.12}$$

obtaining a matrix $P = [p_{ij}]$, whose row vectors are the eigenvectors of A. Then the $\hat{x}$ coordinates are defined by

$$x_j = \sum_{i=1}^{n} p_{ij}\hat{x}_i, \qquad \text{or} \qquad x = P^t\hat{x} \tag{3.3.13}$$

In the new coordinates the differential equation become:

$$\hat{x}_i' = \lambda_i \hat{x}_i \tag{3.3.14}$$

which has the solution

$$\hat{x}_i = a_i e^{\lambda_i t} \tag{3.3.15}$$

The general solution to the original equations found from (3.3.13) and (3.3.15):

$$x_j(t) = \sum_{i=1}^{n} p_{ij} a_i e^{\lambda_i t}, \qquad \text{or} \qquad x(t) = P^t\hat{x}(t) \tag{3.3.16}$$

To find the solution we solve for the initial conditions $x(0)=u=P^t a$ so $a=(P^t)^{-1}u$.

Note that (3.3.16) can also be written

$$x = \sum_{i=1}^{n} a_i e^{\lambda_i t}\hat{e}_i \tag{3.3.17}$$

and it is easy to check that this is a solution directly: Since it is a linear combination it follows that (3.3.17) is a solution if $e^{\lambda_i t}\hat{e}_i$ is a solution, which follows since

$$\frac{d}{dt}\big(e^{\lambda_i t}\hat{e}_i\big) = \lambda_i e^{\lambda_i t}\hat{e}_i, \qquad \text{and} \qquad A\big(e^{\lambda_i t}\hat{e}_i\big) = e^{\lambda_i t}A\hat{e}_i = e^{\lambda_i t}\lambda_i\hat{e}_i \tag{3.3.18}$$

Note that if the eigenvalues are distinct then one knows that the solution is of the general form (3.3.17) and from this, even without having to calculate the eigenvectors, one can conclude certain important qualitative information about the linear system. One can for example conclude if the system is asymptotically stable, which means that the solutions tend to 0 as time t tend to infinity. In particular if all the eigenvalues are negative then the solution (3.3.17) tend to 0 as $t \to \infty$.

What happens if there isn't n distinct real eigenvalues. Well, n distinct real eigenvalues imply that there is a basis of n linearly independent eigenvectors. However, it can happen that we don't have n distinct eigenvalues but we still have n linearly independent eigenvectors. The simplest example if if the matrix is the identity matrix. Then all the eigenvalues are equal to one but any vector is an eigenvectors so we can in particular choose the n vectors to be the unit vectors in the coordinate directions. However, even if we don't have n distinct eigenvectors it could still be that we can can find a solution of a similar form to (3.3.17), which apart from $e^{\lambda_i t}$ also contains terms of the form $te^{\lambda_i t}$.