Lecture 8: Section 3.3. Ex. Find the general solution to

$$\begin{aligned} x_1' &= x_1 \\ x_2' &= x_1 + 2x_2 \\ x_3' &= x_1 - x_3 \end{aligned} \tag{3.3.19}$$

The corresponding matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \tag{3.3.20}$$

The characteristic polynomial is simple to calculate since A is triangular:

$$\text{Det}(A - \lambda I) = (1-\lambda)(2-\lambda)(-1-\lambda) \tag{3.3.21}$$

Since the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = -1$ are real and distinct we know (by section 3.2) that there is matrix Q such that

$$\hat{A} = QAQ^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3.3.22}$$

and in the new coordinates (section 3.1.D) $\hat{x} = Qx$ (3.3.19) become

$$\begin{aligned} \hat{x}_1' &= \hat{x}_1 & & & \hat{x}_1 &= a_1 e^t \\ \hat{x}_2' &= 2\hat{x}_2 & & \text{so} & \hat{x}_2 &= a_2 e^{2t} \\ \hat{x}_3' &= -\hat{x}_3 & & & \hat{x}_3 &= a_3 e^{-t} \end{aligned} \tag{3.3.23}$$

However, in order to find $x = Q^{-1}\hat{x}$, we must find the eigenvectors as well

$$(A - \lambda_i I)\hat{e}_i = 0, \qquad i = 1, 2, 3 \tag{3.3.25}$$

These can be found to be

$$\hat{e}_1 = (2,-2,1) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \quad \hat{e}_2 = (0,1,0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{e}_3 = (0,0,1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.3.26}$$

Note that vectors $(x_1, x_2, x_3) \in \mathbf{R}^3$ are not to be confused with row vectors $[a_{11}\ a_{12}\ a_{13}]$ in matrix notation, but all vectors $(x_1, x_2, x_3) \in \mathbf{R}^3$ are to be thought of as column vectors. We now form the matrix whose column vectors are the eigenvectors

$$P^t = \begin{bmatrix} \hat{e}_1\ \hat{e}_2\ \hat{e}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{3.3.27}$$

and we know from section 3.1.C that $x = P^t \hat{x}$, i.e.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 e^t \\ a_2 e^{2t} \\ a_3 e^{-t} \end{bmatrix} = a_1 e^t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + a_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3 e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.3.28}$$

However (3.3.28) can be seen to be a solution directly. In fact, since it is a linear combination $+a_1 e^t \hat{e}_1 + a_2 e^{2t} \hat{e}_2 + a_3 e^{-t} \hat{e}_3$ of $e^t \hat{e}_1$, $e^{2t} \hat{e}_2$ and $e^{-t} \hat{e}_3$ it sufficies to prove that each of these guys are solutions, i.e. $e^{\lambda_i t} \hat{e}_i$ However, this we follows since $d(e^{\lambda_i t} \hat{e}_i)/dt = \lambda_i e^{\lambda_i t} \hat{e}_i$ and $A(e^{\lambda_i t} \hat{e}_i) = e^{\lambda_i t} A \hat{e}_i = e^{\lambda_i t} \lambda_1 \hat{e}_i$ so they satisfy $x' = Ax$.

Ex. The system

$$x_1' = x_1, \quad x_2' = x_1 + x_2 \tag{3.3.29}$$

has a double eigenvalue 1 and there is only one linearly independent eigenvector $(0, 1)$ belonging to this eigenvalue. It is easy to check that the solution is

$$x_1 = ae^t, \qquad x_2 = e^t(at + b) \tag{3.3.30}$$

Section 3.4: Complex Eigenvalues. Let $T_{a,b}$ be the operator with matrix

$$A_{a,b} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \qquad a, b \in \mathbf{R} \tag{3.4.1}$$

The characteristic polynomial has complex conjugate roots:

$$\lambda^2 - 2a\lambda + (a^2 + b^2) = (\lambda - (a + ib))(\lambda + (a + ib), \qquad i = \sqrt{-1} \tag{3.4.2}$$

We can interpret it geometrically as a rotation in the plane. We can express (a, b) in polar coordinates: $a = r\cos\theta$, $b = r\sin\theta$, where $r = (a^2 + b^2)^{1/2}$. Then $T_{a,b}$ is a counterclockwise rotation through the angle θ followed by a stretching (or shrinking) of the length of the vector by a factor r. I.e. if R_θ denoted the rotation by an angle θ then $T_{a,b}x = rR_\theta(x) = R_\theta(rx)$. In fact the matrix for a rotation is

$$B_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{3.4.3}$$

and the matrix for a multiplication by a scalar is rI and it is easy to check that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{3.4.4}$$

There is another, algebraic, interpretation of the $T_{a,b}$ using complex numbers under the identification $(x, y) \to x + iy$. With this identification the operator corresponds to multiplication with $a + ib$. In fact, $(a + ib)(x + iy) = (ax - by) + (bx + ay)i$ which is the expression for $T_{a,b}(x, y) = (ax - by, bx + ay)$ in the complex plane. Recall also that we can write $a + ib = re^{i\theta}$, where $e^{i\theta} = \cos\theta + i\sin\theta$.

Let us now solve the system

$$x' = ax - by, \qquad y' = bx + ay \tag{3.4.5}$$

Identifying vectors (x, y) with complex numbers $z = x + iy$ we can write this

$$z' = \mu z \tag{3.4.6}$$

where $\mu = a + ib$. The solution for this is

$$z = Ke^{\mu t} \tag{3.4.7}$$

in complex notaion or with $K = u + iv$

$$x + iy = (u + iv)e^{at+ibt} = (u + iv)e^{at}e^{ibt} = (u + iv)e^{at}\big(\cos(bt) + i\sin(bt)\big) \tag{3.4.8}$$

which in real notation is equivalent to

$$x = ue^{at}\cos bt - ve^{at}\sin bt, \qquad y = ue^{at}\sin bt - ve^{at}\cos bt \tag{3.4.9}$$

There are still examples of systems with complex conjugate roots that is not of the above form but that can be transformed to the above form. One is

$$A = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}, \quad \hat{A} = QAQ^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{if} \quad Q = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}. \tag{3.4.10}$$

another one is

$$A = \begin{bmatrix} 0 & 1 \\ -b^2 & 0 \end{bmatrix}, \quad \hat{A} = QAQ^{-1} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \quad \text{if} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}. \tag{3.4.11}$$