Solutions Math 130A Practice Midterm, Winter 2003, Lindblad.

For the solution of the problems two things are good to remember:

The first is how to change coordinates:

$$x_1 e_1 + x_2 e_2 + x_3 e_3 = \hat{x}_1 \hat{e}_1 + \hat{x}_2 \hat{e}_2 + \hat{x}_3 \hat{e}_3$$

from the standard basis $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ to the basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$. If we express $\hat{e}_i$ in terms of the standard basis and form the matrix with these as its column vectors $Q = \begin{bmatrix}\hat{e}_1 \hat{e}_2 \hat{e}_3\end{bmatrix}$ the change of coordinates is given by $x = Q\hat{x}$. In fact, in this case the formula above can be written:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \hat{e}_1 \hat{e}_2 \hat{e}_3 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_n \end{bmatrix}$$

If A is the matrix for the an operator T in the x coordinates, related to the $\hat{x}$ coordinates by $x = Q\hat{x}$, then the matrix for the operator in the $\hat{x}$ coordinates is

$$\hat{A} = Q^{-1} A Q, \qquad \text{and} \qquad A = Q \hat{A} Q^{-1}.$$

This is obvious from the following commuting diagram where the top is the operator expressed in the x coordinates and the bottom row is the operator expressed in the $\hat{x}$ coordinates:

$$\begin{array}{ccc} x & \xrightarrow{A} & Ax \\ {\scriptstyle Q}\uparrow & & \uparrow{\scriptstyle Q} \\ \hat{x} & \xrightarrow{\hat{A}} & \hat{A}\hat{x} \end{array}$$

The operator $\hat{A} : \hat{x} \to \hat{A}\hat{x}$ is defined by first transforming from the $\hat{x}$ coordinates to the x coordinates: $Q : \hat{x} \to x$, then applying $A : x \to Ax$ and finally going back to the $\hat{x}$ coordinates by $Q^{-1} : Ax \to \hat{A}\hat{x}$.

Another thing to remember is Cramer's rule to invert a matrix using sub determinants, see page 325 of the book, since it might go faster. Although, that could be a matter of what one is used to and how many row operations are needed to invert them using Gaussian elimination. A suggestion is to try and see what is fastest. Actually, for me it went faster with row operations and I made less mistakes in doing it that way.

Of course another thing to remember is how to find the general solution of a system of linear algebraic equations. First one transforms the system to triangular form with Gaussian elimination. The triangular system is solved by back substitution, setting any free variable equal to a parameter. In our case, when we are looking for the generalized eigenspaces, there will not be a unique solution but a solution space depending on parameters, and this gives a basis for the solution space. Review your linear algebra book for details.

Lastly, on the actual exam we will not subtract many points if you leave the answer in a form so it is only left to multiply matrices together. Multiplying matrices together takes a long time so your time is better spend on starting on all problems. However, if you have time you should do the multiplications, since in that way you can check that your answer is correct.

1. (a) The eigenvalues are complex but distinct: 3, $2+i\sqrt{3}$ and $2-i\sqrt{3}$. Therefore the matrix is complex diagonalizable, i.e semisimple. Since the decomposition of the operator $T=S+N$, into a semisimple S part and a nilpotent part N that commute, is unique and since we already deduced that A is semisimple it follows that $N=0$. The complex standard form is the diagonal matrix diag$\{3,\ 2+i\sqrt{3},\ 2-i\sqrt{3}\}$ and from this we know that the real standard form is:

$$\hat{A} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & -\sqrt{3} \\ 0 & \sqrt{3} & 2 \end{bmatrix}$$

(b). In order to calculate the exponential matrix we also have to know what the changes of coordinates is that puts A in the above form. To do this we first have to find the complex eigenvectors of A, and from those construct the new basis.

The system $(A-3I)x=0$ can be written

$$\begin{bmatrix} -1 & 0 & -3 \\ -1 & 0 & -7 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Leftrightarrow \qquad \left[\begin{array}{ccc|c} -1 & 0 & -3 & 0 \\ -1 & 0 & -7 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

where the augmented matrix to the right is the shorter way to write the system to the left. Using Gaussian elimination, first adding the last row to the first two, secondly diving the second row by -8, the first by 4 and adding the second to the first and finally adding the second to the last gives gives

$$\left[\begin{array}{ccc|c} 0 & 0 & -4 & 0 \\ 0 & 0 & -8 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

The system now says that $x_1=0$ and $x_3=0$ so $x_2=t$ for any t. Taking $t=1$ gives us the eigenvector $\hat{e}_1=(0,1,0)$.

The system $(A-i\sqrt{3}I)x=0$ is equivalent to

$$\left[\begin{array}{ccc|c} -i\sqrt{3} & 0 & -3 & 0 \\ -1 & 1-i\sqrt{3} & -7 & 0 \\ 1 & 0 & -i\sqrt{3} & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1-i\sqrt{3} & -7-i\sqrt{3} & 0 \\ 1 & 0 & -i\sqrt{3} & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & -1-i2\sqrt{3} & 0 \\ 1 & 0 & -i\sqrt{3} & 0 \end{array}\right]$$

since $\dfrac{-7-i\sqrt{3}}{1-i\sqrt{3}} = \dfrac{-7-i\sqrt{3}}{1-i\sqrt{3}}\dfrac{1+i\sqrt{3}}{1+i\sqrt{3}} = \dfrac{-4-i8\sqrt{3}}{4} = -1-2i\sqrt{3}$. This says that $x_1-i\sqrt{3}x_3=0$ and $x_2-(1+i\sqrt{3})x_3=0$. If we set $x_3=t=1$ we hence get $x=(i\sqrt{3},1+2i\sqrt{3},1)=(0,1,1)+i(\sqrt{3},2\sqrt{3},0)$. We therefore put $\hat{e}_3=(0,1,1)$ i.e. the real part of x and $\hat{e}_2=(\sqrt{3},2\sqrt{3},0)$, the imaginary part

We now form the matrix

$$Q = \begin{bmatrix} \hat{e}_1\hat{e}_2\hat{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{3} & 0 \\ 1 & 2\sqrt{3} & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \cdots \Rightarrow Q^{-1} = \begin{bmatrix} -2 & 1 & -1 \\ 1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We know that $A=Q\hat{A}Q^{-1}$, see previous page. Hence it follows that

$$e^{At} = Qe^{\hat{A}t}Q^{-1} = \begin{bmatrix} 0 & \sqrt{3} & 0 \\ 1 & 2\sqrt{3} & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t}\cos\sqrt{3}t & -e^{2t}\sin\sqrt{3}t \\ 0 & e^{2t}\sin\sqrt{3}t & e^{2t}\cos\sqrt{3}t \end{bmatrix} \begin{bmatrix} -2 & 1 & -1 \\ 1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that in order to get exactly the matrix $\hat{A}$ above we must have that $\hat{e}_3+i\hat{e}_2$ is the eigenvector corresponding to the eigenvalue $2+i\sqrt{3}$. Interchanging $\hat{e}_2$ and $\hat{e}_3$ changes the sign in front of $\sqrt{3}$.

2.(a) The eigenvalues are 3, 9 and 9. We solve $(A-3I)x=0$:

$$\left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ -2 & 2 & 2 & 0 \\ -12 & -12 & 12 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

i.e. $x_2=0$ and $x_1-x_3=0$ so $x=(t,0,t)$ and we take $\hat{e}_1=(1,0,1)$.

The eigenvalue 9 is double which means that there is a two dimensional generalized eigenspace to $(A-9I)^2x=0$. (We could have tried to first see if there is a two dimensional space of solutions to $(A-9I)x=0$ and if so this would also be the two dimensional solution space to $(A-9I)^2x=0$ so we already have our two basis vectors $\hat{e}_2$ and $\hat{e}_3$ for the generalized eigenspace. On the other hand, if there was only a one dimensional solution space to $(A-9I)x=0$ we get one basis vector $\hat{e}_2$ in this way and we can then go on solving $(A-9I)x=\hat{e}_2$ to find another basis vector $\hat{e}_3$. It is then clear that $(A-9I)^2\hat{e}_2=(A-9I)(A-9I)\hat{e}_2=(A-9I)0=0$ and $(A-9I)^2\hat{e}_3=(A-9I)(A-9I)\hat{e}_3=(A-9I)\hat{e}_2=0$.) However, instead of doing this we directly start solving for the two dimensional generalized eigenspace:

$$(A-9I)=\begin{bmatrix} 8 & -4 & 2 \\ -2 & -4 & 2 \\ -12 & -12 & 6 \end{bmatrix} \Rightarrow \cdots \Rightarrow (A-9I)^2=\begin{bmatrix} 48 & 24 & -12 \\ 0 & 0 & 0 \\ 48 & 24 & -12 \end{bmatrix}$$

We solve

$$\left[\begin{array}{ccc|c} 48 & 24 & -12 & 0 \\ 0 & 0 & 0 & 0 \\ 48 & 24 & -12 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 4 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

i.e $4x_1+2x_2-x_3=0$. Hence the general solution is $(x,y,z)=(t,s,4t+2s)$ so we can take $\hat{e}_2=(1,0,4)$ and $\hat{e}_3=(0,1,2)$.

We now want to make the change of variables $x=Q\hat{x}$, where

$$Q=\begin{bmatrix} \hat{e}_1\hat{e}_2\hat{e}_3 \end{bmatrix}=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix} \Rightarrow \cdots \Rightarrow Q^{-1}=\begin{bmatrix} 4/3 & -2/3 & 1/3 \\ -1/3 & -2/3 & 1/3 \\ 0 & 1 & 0 \end{bmatrix}$$

In this case our original matrix A is not semisimple (Because we do not have 3 linearly independent eigenvectors; one can check that for the double eigenvalue there is only one linearly independent eigenvector.) However, if we let

$$\hat{S}=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \quad S=Q\hat{S}Q^{-1}=\cdots=\begin{bmatrix} 1 & -4 & 1 \\ 0 & 9 & 0 \\ -4 & -2 & 10 \end{bmatrix}, \quad N=A-S=\begin{bmatrix} 0 & 0 & 0 \\ -2 & -4 & 2 \\ -4 & -8 & 4 \end{bmatrix}$$

One can now check that $N^2=0$ and $SN=NS$, but this follows from the theory. In fact, we can write $E=E_1 \oplus E_2$, where E_1 is the eigenspace belonging to the eigenvalue 3, spanned by $\hat{e}_1$, and E_2 is the generalized eigenspace belonging to the eigenvalue 9, spanned by $\hat{e}_2$ and $\hat{e}_3$. Since A leaves E_1 and E_2 invariant it follows that $\hat{A}=Q^{-1}AQ=\text{diag}\{\hat{A}_1,\hat{A}_2\}$ and $\hat{N}=\hat{A}-\hat{S}=\text{diag}\{\hat{N}_1,\hat{N}_2\}$ consist of a 1×1 block in the upper left corner and a 2×2 block in the lower right corner, and zeros elsewhere. $\hat{N}_1$ is the matrix for $A-3I|_{E_1}$ so $\hat{N}_1=0$ and $\hat{N}_2$ is the matrix for $(A-9I)|_{E_2}$ so $\hat{N}_2^2=0$. Hence $\hat{N}^2=\text{diag}\{\hat{N}_1^2,\hat{N}_2^2\}=0$ and $\hat{N}\hat{S}-\hat{S}\hat{N}=\text{diag}\{\hat{N}_1 3I-3I\hat{N}_1,\hat{N}_2 9I-9I\hat{N}_2\}=0$.

(b). Since $A=S+N$, where $SN=NS$ and $S=Q\hat{S}Q^{-1}$, where $\hat{S}$ is diagonal;

$$e^{At}=e^{St+Nt}=e^{St}e^{Nt}=Qe^{\hat{S}t}Q^{-1}(I+tN+t^2N^2/2), \quad e^{\hat{S}t}=\begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{9t} & 0 \\ 0 & 0 & e^{9t} \end{bmatrix}$$