Lecture 19: 7.1: Hyperbolic flows. A type of linear flow e^{tA} that is more general than contractions and expansions is the *hyperbolic flow*, i.e. when all eigenvalues have nonzero real part.

Theorem. *Let e^{tA} be a hyperbolic linear flow, where $A \in L(E)$, the linear operators on E. Then E has a direct sum decomposition: $E = E^s \oplus E^u$, invariant under A, such that the induced flow on E^s is a contraction and the induced flow on E^u is an expansion. This decomposition is unique.*

For the proof we give E a basis putting A in real canonical form. E^s is then spanned by all the basis vectors corresponding to the generalized eigenspaces of eigenvalues with negative real part and E^u is then spanned by all the basis vectors corresponding to the generalized eigenspaces of eigenvalues with positive real part.

E^s is called the *stable* subspace and E^u is called the *unstable* subspace.

7.3: Generic properties of operators. Let F be a normed vector space. The subset $\{y \in F; |y - x| < a\}$ is called an *open ball of radius a about x*. A subset $X \subset F$ is called *open* if, for every $x \in X$, there is an $a > 0$ such that the open ball of radius a about x is contained in F. Loosely speaking, if x is in F then every point sufficiently close to x is also in F. Open sets are therefore "thick". E.g. a surface in space is too thin to be open, since for any point on it there are points outside arbitrarily close. A subset $X \in F$ is called *dense*, if for every $x \in F$ and $\varepsilon > 0$ there is a $y \in X$ such that $|x - y| < \varepsilon$. Loosely speaking, a set X is dense if every point in F is arbitrarily close to points in X. We will consider sets that are both dense and open, and in a sense rather filling.

Now, consider a subset X of the normed vector space $L(\mathbf{R}^n)$. A property $\mathcal{P}$ that refers to operators on $\mathbf{R}^n$ is a *generic property* if the set of operators having the property $\mathcal{P}$ contains a dense open set. We say that a property hold for *almost all* operators if it is a generic property. The main theorem in this section is:

Theorem 1. *The set $\mathcal{S}_1$ of operators on $\mathbf{R}^n$ that have n distinct eigenvalues is dense and open in $L(\mathbf{R}^n)$.*

Proof. Given an operator T we first want to find an operator arbitrarily close with distinct eigenvalues to prove that these operators are dense. Find a basis $\mathcal{B}$ that puts $T = S + N$ in real canonical form, where $S = \text{diag}\{\lambda_1, ..., \lambda_p, A_{p+1}, ..., A_m\}$, where $A_k = \begin{bmatrix} a_k & -b_k \\ b_k & a_k \end{bmatrix}$ and N is in nilpotent standard form. For any $\varepsilon > 0$ we can find distinct numbers $\lambda_1', ..., \lambda_p', a_{p+1}', ..., a_m'$ such that $|\lambda_k - \lambda_k'| < \varepsilon$, for $k = 1, ..., p$ and $|a_k - a_k'| < \varepsilon$, for $k = p+1, ..., m$. Set $S' = \text{diag}\{\lambda_1', ..., \lambda_p', A_{p+1}', ..., A_m'\}$, where $A_k' = \begin{bmatrix} a_k' & -b_k \\ b_k & a_k' \end{bmatrix}$. Then $T' = S' + N$ has distinct eigenvalues and the norm $\|S - S'\|_{\mathcal{B}} = sup_{|x|_{\mathcal{B}} \leq 1} |(S - S')x|_{\mathcal{B}} \leq \varepsilon$. For the proof of openness see the book. □

Theorem 2. *Semisimplicity is a generic property.*

Theorem 3. *The set of operators with hyperbolic flow is an open dense set in $L(\mathbf{R}^n)$.*

Many things that are true for semisimple matrices are now true in general:

Cayley-Hamilton Theorem. *If $p(t) = a_0 + ... + a_n t^n$ is the characteristic polynomial for A, then $p(A) = a_0 + ... + a_n A^n = 0$.*

In fact, there is a sequence A_j such that $\|A - A_j\| \leq 1/j$ and $A_j = Q\hat{A}_j Q^{-1}$, where $\hat{A}_j = \text{diag}\{\lambda_1, ..., \lambda_n\}$. Then $p_j(A) = Qp_j(\hat{A})Q^{-1} = Q\,\text{diag}\{p(\lambda_1), ..., p(\lambda_n)\}Q^{-1} = 0$.