

Lecture 24: 9.3 Cont..

Theorem 2. Suppose that V is a Liapunov function in a neighborhood U of the equilibrium point \bar{x} of the system $x' = f(x)$. Suppose also that V is not constant on any solution curve in U other than \bar{x} itself. Then \bar{x} is asymptotically stable.

Let $U_\alpha = \{x \in U; V(x) < \alpha\}$ and suppose that the closure \bar{U}_α is a bounded set contained in U . Then any solution curve starting in U_α converges to \bar{x} .

Proof. First, U_α is a neighborhood of \bar{x} such that any solution curve starting in U_α is contained in U_α . If $\alpha > 0$ is sufficiently small then $\bar{U}_\alpha \subset U$. Suppose that $x(t)$ is a solution curve in U_α that does not tend to \bar{x} . Then by the proof of Theorem 1 there must exist a sequence such that $x(t_n) \rightarrow a \neq \bar{x}$ and $V(x(t))$ must decrease to $V_0 > 0$ as $t \rightarrow \infty$. Let $z(s)$ be the solution to $x' = F(x)$ with initial data $z(0) = a$. Note that $x_n(s) = x(t_n + s)$ are solutions with initial data $x(t_n)$. It follows that $x_n(0) \rightarrow z(0) = a$ and hence by continuous dependence on initial conditions, see section 8.4, $x_n(s) \rightarrow z(s)$ and it follows that $V(z(s)) = \lim_{n \rightarrow \infty} V(x(t_n + s)) = \lim_{t \rightarrow \infty} V(x(t)) = V_0$. Therefore V is constant on the entire orbit of a which by assumption implies that $a = \bar{x}$. \square

Ex. Show that the equilibrium corresponding to the pendulum hanging still and straight down is asymptotically stable.

Sol. Recall the equations for the pendulum from section 9.1:

$$(9.3.9) \quad (\theta', \omega') = f(\theta, \omega), \quad \text{where } f(\theta, \omega) = \left(\omega, -\frac{k}{m}\omega - \frac{1}{\ell} \sin \theta \right)$$

and that $(\theta, \omega) = (0, 0)$ is an equilibrium. An energy is given by

$$(9.3.10) \quad E = \text{kinetic energy} + \text{potential energy} = \frac{1}{2}mv^2 + m(\ell - \ell \cos \theta)$$

and hence

$$(9.3.11) \quad E(\theta, \omega) = \frac{1}{2}m\ell^2\omega^2 + m\ell(1 - \cos \theta)$$

Then

$$(9.3.12) \quad \dot{E}(\theta, \omega) = m\ell(\ell\omega' + \sin \theta \theta') = -k\ell^2\omega^2$$

Thus $\dot{E} \leq 0$ and $E(0, 0) = 0$ so E is a Liapunov function and it follows from Theorem 1 that $(0, 0)$ a stable equilibrium. Since E vanishes when $\omega = 0$ it is however not a strict Liapunov function. Instead we will prove that the condition in Theorem 2 hold. Suppose that E is constant along an entire orbit. Then $\omega = 0$ on that orbit so θ is constant but since ω is constant along the entire orbit it follows that $\omega' = 0$ which implies that $\sin \theta = 0$ as well so $\theta = 0$ or $\theta = \pm\pi$. Therefore $(\theta, \omega) = (0, 0)$ is the only entire orbit on which E is constant in the set $U = \{(\theta, \omega); |\theta| < \pi\}$. Now, if $\alpha < 2m\ell$ then the closure of $U_\alpha = \{(\theta, \omega) \in U; E(\theta, \omega) < \alpha\}$, is contained in U , for then $E(\theta, \omega) \leq \alpha$ and hence $\cos \theta \geq -\alpha/(2m\ell) > -1$. Note that it follows from the proof of Theorem 1 and Theorem 2 that U_α , for $\alpha < 2m\ell$ are *positively invariant*, i.e. any solution curve starting in U_α will remain in U_α , because E is decreasing and leaving U_α , E would have to increase over the value on the boundary of U which is $2m\ell$.