Section 4.1: Complex vector spaces. The *Complex Cartesian space* $\mathbf{C}^n$ is the set of n-tuples $z=(z_1,...,z_n)$ of complex numbers $z_i \in \mathbf{C}$. Since a complex number $x+iy \in \mathbf{C}$ can be identified with a point in the plane $(x,y) \in \mathbf{R}^2$ we can identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$. Complex vectors can be added just like vectors is $\mathbf{R}^{2n}$, i.e. if also $w=(w_1,...,w_n) \in C^n$ then $z+w=(z_1+w_1,...,z_n+w_n)$. However, there is an additional structure which is scalar multiplication with a complex number. If $\lambda \in \mathbf{C}$ then $\lambda z = (\lambda z_1,...,\lambda z_n)$. The vector space axioms of section 3.1 holds for $\mathbf{C}^n$ with the addition and scalar multiplication we just defined and they define what we call a *complex vector space structure* on $\mathbf{C}^n$.

A subset $F \subset \mathbf{C}^n$ is called a subspace or a (complex) linear subspace if it is closed under addition and scalar multiplication, compare (3.1.9) with $\mathbf{R}$ replaced by $\mathbf{C}$:

$$(4.1.1) \qquad x,y \in F, \qquad \lambda \in \mathbf{C} \qquad \Longrightarrow \qquad x+y \in F, \qquad \lambda x \in F$$

By a *Complex vector space* we will mean a complex subspace of $\mathbf{C}^n$.

Ex. A line in the x-y plane through the origin, e.g the real line $y=0$, is a real subspace of the plane $\mathbf{R}^2$, but it is not a complex subspace of $\mathbf{C}$ since if we multiply a number on the real line by a complex number we do not get a real number anymore.

A (complex) linear map $T: F_1 \to F_2$ or complex operator from one complex subspace to another is a map such that

$$(4.1.2) \qquad T(x+y)=Tx+Ty, \qquad T(\lambda x)=\lambda Tx, \qquad x,y \in F_1, \qquad \lambda \in \mathbf{C}$$

A complex basis $\{e_i,...,e_n\}$ is a set of vectors $e_i \in \mathbf{C}^n$ such that we can write any vector in a unique way as linear combination of these with complex coefficients:

$$(4.1.3) \qquad z=\sum_{i=1}^n t_i e_i, \qquad t_i \in \mathbf{C}$$

Everything in section 3.1 goes through with the real vector spaces replaced by complex vector spaces, scalar multiplication by a real number replaced by scalar multiplication by a complex number and the real operators replaced by complex operators. The complex dimension of a complex vector space, the kernel and image of a complex linear operator and the determinant are defined in exactly the same way as in the real case. Most importantly, for a complex $n \times n$ matrix A:

$$(4.1.4) \qquad \text{Det}\, A = 0 \Leftrightarrow Ax=0 \text{ for some complex vector } x \neq 0.$$

The reader should convince herself/himself that this is true.

We define a complex eigenvector $v \in \mathbf{C}^n$ to be a nonzero vector such that for some complex eigenvalue $\lambda \in \mathbf{C}$:

$$(4.1.5) \qquad Tv=\lambda v$$

and we define the characteristic polynomial to be

$$(4.1.6) \qquad p(\lambda)=\text{Det}(T-\lambda I)$$

By (4.1.4), λ is an eigenvalue, i.e. (4.1.5) has a nontrivial solution, if and only if it is a root of the characteristic polynomial $p(\lambda)=0$. The proof of Theorem 3.2.1 gives:

Theorem. *Let $T : F \to F$ be an operator on an n-dimensional complex vector space F. If the characteristic polynomial has distinct roots, then T can be diagonalized, i.e. one can find a basis $\{\hat{e}_1, ..., \hat{e}_n\}$ of eigenvectors of T so that if $z = \sum_{j=1}^n z_j \hat{e}_j$ then $Tz = \sum_{j=1}^n \lambda_j z_j \hat{e}_j$, where λ_i is the eigenvalue corresponding to e_i.*

Let F be a complex subspace of $\mathbf{C}^n$ and let $F_{\mathbf{R}}$ be the *space of real vectors* of F:

$$F_{\mathbf{R}} = F \cap \mathbf{R}^n \tag{4.1.7}$$

Let E be a subspace of $\mathbf{R}^n$ and let $E_{\mathbf{C}}$ be the *complexification* of E, i.e. the complex subspace of $\mathbf{C}^n$ obtained by taking all linear combinations of vectors in E with complex coefficients:

$$E_{\mathbf{C}} = \{z \in \mathbf{C}^n : z = \sum_{i=1}^k \lambda_i z_i, \; z_i \in E\} \tag{4.1.8}$$

Note that $(E_{\mathbf{C}})_{\mathbf{R}} = E$.

Recall that if $z = x+iy \in \mathbf{C}$ then the *complex conjugate* is defined by $\overline{z} = x-iy \in \mathbf{C}$. Let us denote the map taking a complex number to its complex conjugate $\overline{z}$ by $\sigma : \mathbf{C} \to \mathbf{C}$. Note that $\sigma \circ \sigma(z) = z$ and $\sigma(z) = z$ if and only if z is real. We define

$$\sigma(z_1, ..., z_n) = (\overline{z}_1, ..., \overline{z}_n) \tag{4.1.9}$$

It follows that $F_{\mathbf{R}} = \{z \in F : \sigma(z) = z\}$. We note that a subspace $F \in \mathbf{C}^n$ can be decomplexified, i.e. written as $F = E_{\mathbf{C}}$ for some real vector space E if and only if $\sigma(F) \subset F$. In fact, if $\sigma(F) \subset F$ then $x - iy \in F$ whenever $x + iy \in F$ so $x = (x + iy)/2 + (x - iy)/2 \in F$ and $y = (x + iy)/2 - (x - iy)/2 \in F$. It follows that $F = (F_{\mathbf{R}})_{\mathbf{C}}$. On the other hand $\sigma(E_{\mathbf{C}}) \subset E_{\mathbf{C}}$.

Just as a real subspace has a complexification a real operator $T : E \to E$ has a complexification $T_C : E_{\mathbf{C}} \to E_{\mathbf{C}}$ defined by

$$T_C\left(\sum \lambda_i z_i\right) = \sum \lambda_i T z_i, \qquad z_i \in E, \qquad \lambda_i \in \mathbf{C} \tag{4.1.10}$$

If T is represented by A then T_C is also represented by A. When is an operator $Q : E_C \to E_C$ the complexification of an operator $T : E \to E$?

Theorem. *Let E be a real vector space and E_C its complexification. If Q is a a complex linear operator on E_C then $Q = T_C$ is the complexification of some real operator T on E if and only if $Q\sigma = \sigma Q$, where $\sigma : E_C \to E_C$ is the conjugation.*

Proof. First if Q commutes with conjugation then $Q(E) \subset E$, since if $x \in E$ $\sigma Q x = Q \sigma x = Q x$ so $Qx \in (E_{\mathbf{C}})_{\mathbf{R}}$. Let $T = Q\big|_E$ $\square$