Solutions to Homework 9

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Problem 8.1: For the non-linear systems, $x' = x + y^2$, $y' = 2y$; and $x' = x^2$, $y' = y^2$,

1. Find all equilibrium points and describe the behaviour of the linearised equation at these points.
2. Describe the phase-portrait of the non-linear system.
3. Does the linearised system accurately describe the local behaviour of the non-linear system near the equilibrium points?

Solution: Let us first consider the system

$$\begin{aligned} x' &= x + y^2, \\ y' &= 2y. \end{aligned}$$

1. We have equilibrium points at (x, y) with $0 = x + y^2$ and $0 = 2y$, that is to say, where $x = 0$ and $y = 0$. The linearised system at $(0, 0)$ is the original system with the non-linear terms omitted, so the linearised system is

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & 2v \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

   which has solution

$$c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

   that is, the point $(0, 0)$ is a source.

2. The non-linear system can also be solved explicitly. First, we solve for y in terms of t and find $y(t) = y_0 e^{2t}$, where $y(0) = y_0$. Then we solve for x by substituting the above solution for y into the equation for x'. From $x' = x + y_0^2 e^{4t}$, we obtain the solution

$$x(t) = \frac{y_0^2 e^{4t}}{3} + \left(x_0 - \frac{y_0^2}{3} \right) e^t,$$

   where $x(0) = x_0$.

3. Near the equilibrium point we can suppose that the initial condition (x_0, y_0) is near $(0, 0)$. This means that y_0^2 is negligible. Thus near the equilibrium point the solution is approximately,

$$\begin{aligned} x &= x_0 e^t, \\ y &= y_0 e^{2t}, \end{aligned}$$

   which coincides with the solution to the linearised system.

Let us now consider the system

$$\begin{aligned} x' &= x^2, \\ y' &= y^2. \end{aligned}$$

1. We have equilibrium points at (x, y) with $0 = x^2$ and $0 = y^2$, so there is only one equilibrium point, and it is at the origin. The linearised system at $(0, 0)$ is

$$\begin{aligned} u' &= 0 \\ v' &= 0, \end{aligned}$$

which has solution

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

2. We can solve the non-linear system explicitly in this case as well. We have

$$\begin{aligned} \frac{dx}{dt} &= x^2 \\ \frac{dy}{dt} &= y^2, \end{aligned}$$

from which we obtain by integration

$$\begin{aligned} \frac{-1}{x} + c_1 &= t \\ \frac{-1}{y} + c_2 &= t. \end{aligned}$$

The constants can be determined using the initial values. If $0 = x(0) = y(0)$, then the solution is simply $(x, y) = (0, 0)$. If $x(0) =0$ and $y(0) = y_0 \neq 0$, then

$$\begin{aligned} x(t) &= 0 \\ y &= \frac{-y_0}{1 + y_0 t}, \end{aligned}$$

and similarly, if $x(0) \neq 0$ and $y(0) = 0$. If both $x_0 = x(0)$ and $y_0 = y(0)$ are non-zero, then we have the solution curve given by

$$y = \frac{x}{1 - cx},$$

where $c = \frac{1}{x_0} - \frac{1}{y_0}$.

3. The behaviour of the linear system does not describe the behaviour of the non-linear system accurately, even locally.

Problem 8.2: Find a global change of coordinates which linearises the system

$$\begin{aligned} x' &= x + y^2 \\ y' &= -y \\ z' &= -z + y^2. \end{aligned}$$

Solution: Because the y part of the system is decoupled, we can solve for y to obtain

$$y(t) = y_0 e^{-t},$$

where $y_0 = y(0)$, as usual. We can then substitute $y(t) = y_0 e^{-t}$ into the equation which gives x'. Thus we have

$$x' = x + y_0^2 e^{-2t},$$

which we can solve to obtain

$$x(t) = \frac{-y_0^2 e^{-2t}}{3} + \left(x_0 + \frac{y_0^2}{3}\right) e^t,$$

where $x_0 = x(0)$. Finally, we substitute $y(t) = y_0 e^{-t}$ into the equation which gives z', to obtain

$$z(t) = -y_0^2 e^{-2t} + (z_0 + y_0^2) e^{-t}.$$

From this we see that if $x_0 + \frac{y_0^2}{3} = 0$ and $z_0 + y_0^2 = 0$, we have $x(t) + \frac{y(t)^2}{3} = 0$ and $z(t) + y(t)^2 = 0$, for all t, so these two curves are stable, and we should try to map them to the x and z axes respectively. Define

$$\begin{aligned} u(t) &= x(t) + \frac{y(t)^2}{3} \\ v(t) &= y(t) \\ w(t) &= z(t) + y(t)^2. \end{aligned}$$

Then we have

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

which is linear.

Problem 8.5: Consider the system

$$\begin{aligned} x' &= x^2 + y \\ y' &= x - y + a, \end{aligned}$$

where a is a real parameter.

1. Find all equilibrium points and compute the linearised system at each equilibrium point.
2. Describe the behaviour of the linearised system at each equilibrium point.
3. Describe any bifurcations which occur.

Solution:

1. We will have an equilibrium point at (x, y) if $0 = x^2 + y$ and $0 = x - y + a$; that is, if $-x^2 = x + a$. Thus (x, y) is an equilibrium point if

$$x = \frac{-1 - \sqrt{1 - 4a}}{2} \text{ or } x = \frac{-1 + \sqrt{1 - 4a}}{2}.$$

Clearly, there is bifurcation at $a = \frac{1}{4}$, since if $a < \frac{1}{4}$, then $0 < 1 - 4a$, which means that there are two equilibrium points; if $a = \frac{1}{4}$, then there is one equilibrium point, and if $a > \frac{1}{4}$, then $0 > 1 - 4a$, which means that there are none. Define $f_1(x, y) = x^2 + y$ and define $f_2(x, y) = x - y + a$. Using these we define

$$F(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix},$$

and thus we have

$$\begin{aligned} D(F) &= \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} 2x & 1 \\ 1 & -1 \end{pmatrix}. \end{aligned}$$

Suppose that $a \leq \frac{1}{4}$. Then at the equilibrium point when $x = \frac{-1-\sqrt{1-4a}}{2}$ we have

$$D(F)\Big|_{x=\frac{-1-\sqrt{1-4a}}{2}} = \begin{pmatrix} -1-\sqrt{1-4a} & 1 \\ 1 & -1 \end{pmatrix}.$$

Call the above matrix $A(a)$. Note that $A(a)$ is only defined for $a \leq \frac{1}{4}$. The following idea is due to Brianna Cardiff: We have

$$\text{tr}A(a) = -2-\sqrt{1-4a} \text{ and } \det A(a) = \sqrt{1-4a}.$$

so the matrix $A(a)$ traces out the trajectory

$$\Big\{(D,T) \in \mathbf{R}^2 : D = -T-2, T \leq -2\Big\},$$

in the trace-determinant plane. So, for $a = \frac{1}{4}$, the linearised system around the equilibrium point when $x = \frac{-1-\sqrt{1-4a}}{2} = \frac{-1}{2}$ has one zero eigenvalue and one negative eigenvalue. For $a < \frac{1}{4}$, the linearised system around the equilibrium point when $x = \frac{-1-\sqrt{1-4a}}{2}$ is a sink.

Again, note that we must have $a \leq \frac{1}{4}$. At the equilibrium point when $x = \frac{-1+\sqrt{1-4a}}{2}$ we have

$$D(F)\Big|_{x=\frac{-1+\sqrt{1-4a}}{2}} = \begin{pmatrix} -1+\sqrt{1-4a} & 1 \\ 1 & -1 \end{pmatrix}.$$

Let $B(a)$ denote the above matrix.

$$\text{tr}B(a) = -2+\sqrt{1-4a} \text{ and } \det B(a) = -\sqrt{1-4a}.$$

so the matrix $B(a)$ traces out the trajectory

$$\Big\{(D,T) \in \mathbf{R}^2 : D = -T-2, T \geq -2\Big\},$$

in the trace-determinant plane. So, for $a = \frac{1}{4}$, the linearised system around the equilibrium point when $x = \frac{-1-\sqrt{1-4a}}{2} = \frac{-1}{2}$ has one zero eigenvalue and one negative eigenvalue. For $a < \frac{1}{4}$, the linearised system around the equilibrium point when $x = \frac{-1-\sqrt{1-4a}}{2}$ is a saddle.

Please note that the next two problems are no longer part of the homework. But I had already typed them up, so why not include them. I did not, however, proof-read my solutions so there may be errors. Of course, this goes for all my solutions, but they are probably more likely to occur in the next two solutions.

Problem 8.6: Let $x' = f_a(x)$, and suppose that there are no equilibrium points when $a < 0$; that there is a single equilibrium point when $a = 0$; and that there are four equilibrium points when $a > 0$. Sketch the bifurcation diagram for this family of differential equations.

Solution: The sketches are on a separate sheet of paper.

Problem 8.9: Consider the system

$$\begin{aligned} r' &= r(1-r) \\ \theta' &= \sin\theta + a, \end{aligned}$$

where a is a real parameter.

1. For which values of a does this system undergo bifurcation?
2. Describe the local behaviour of the solutions near the bifurcation values.
3. Sketch the phase portrait for all possible cases.
4. Discuss any global changes which occur at the bifurcations.

Solution:

1. There is always going to be an equilibrium point when $r = 0$, so the bifurcations are going to occur when other equilibrium points appear. For $a < -1$ we have $\theta' = \sin\theta + a < 0$, so there can be no other equilibrium points in this case. Similarly, for $a > 1$ we have $\theta' = \sin\theta + a > 0$, so there will be no other equilibrium points in this case either. For $a = -1$, we have $\theta'(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) - 1 = 0$, so now we have the additional equilibrium point $(r, \theta) = (1, \frac{\pi}{2})$. For $-1 < a < 0$, we have θ_1 and θ_2, with $0 < \theta_1 < \theta_2 < \pi$ for which $\theta'(\theta_1) = \theta'(\theta_2) = 0$, so now we have two additional equilibrium points $(1, \theta_1)$, and $(1, \theta_2)$. For $0 < a < 1$, we have θ_3 and θ_4, with $-\pi < \theta_3 < \theta_4 < 0$ for which $\theta'(\theta_3) = \theta'(\theta_4) = 0$, which gives two additional equilibrium points $(1, \theta_3)$, and $(1, \theta_4)$. So there is bifurcation for $a = -1$ and for $a = 1$.
2. First let us consider what happens near -1. For $a < -1$, we have only the equilibrium point at the origin. We have $\theta' < 0$, and the circle given by $r = 1$ is a periodic solution. For $0 < r < 1$, we have $r' > 0$, so here the solutions spiral counter-clockwise towards the circle; for $1 < r < \infty$, we have $r' < 0$, so here the solutions spiral towards the circle too. When $a = -1$ we have the equilibrium point $(1, \frac{\pi}{2})$. Again, for $0 < r < 1$, we have $r' > 0$, so here the solutions spiral towards the given by $r = 0$; and for $1 < r < \infty$, we have $r' < 0$, so here the solutions spiral towards the circle too, however, the solutions cannot spiral completely around the origin, because once the curve hits the y-axis we have $\theta' = 0$, and thus the solution goes towards the point $(1, \frac{\pi}{2})$. For $-1 < a < 0$, we have the two equilibrium points $(1, \theta_1)$, and $(1, \theta_2)$. For $0 < \theta < \theta_1$, we have $\theta' < 0$; for $\theta_1 < \theta < \theta_2$ $\theta' > 0$; and for $\theta_2 < \theta < \pi$, $\theta' < 0$. Similarly, we can obtain an idea of what happens around 1.
3. The sketches can be found on the attached sheet.

Extra Problem: Use the variation of parameters formula to solve the non-autonomous initial value problem

$$X' = AX + G(t), \quad X(0) = 0,$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$G(t) = \begin{pmatrix} 0 \\ \sin(t) \end{pmatrix}.$$

Solution: The variation of parameter formula says that the solution will be of the form

$$X(t) = \exp(tA) \int_0^t \exp(-sA)G(s)ds.$$

From an example we know that

$$\exp(uA) = \begin{pmatrix} \cos(u) & \sin(u) \\ -\sin(u) & \cos(u) \end{pmatrix},$$

and therefore

$$\exp(-sA)G(s) = \begin{pmatrix} -\sin^2(s) \\ \cos(s)\sin(s) \end{pmatrix}.$$

Thus

$$\begin{aligned} X(t) &= \exp(tA)\int_0^t \exp(-sA)G(s)ds \\ &= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} -\frac{t}{2} + \frac{1}{4}\sin(2t) \\ \frac{1}{4} - \frac{1}{4}\cos(2t) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{t}{2}\cos(t) + \frac{1}{4}\cos(t)\sin(2t) + \frac{1}{4}\sin(t) - \frac{1}{4}\cos(2t)\sin(t) \\ \frac{t}{2}\sin(t) - \frac{1}{4}\sin(t)\sin(2t) - \frac{1}{4}\sin(t) - \frac{1}{4}\cos(t)\cos(2t) \end{pmatrix}. \end{aligned}$$