Math 130B Practice Midterm, Spring 2003, Lindblad.

1. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a continuous function such that $|f(x)| \le M$ and $|f(x) - f(y)| \le L|x - y|$ for all x and y. Consider $x' = f(x)$, where $x(0) = x_0$.
a) Show that there is a local solution, i.e. there is $T > 0$ such that there is a solution $x(t)$ for $0 \le t \le T$, by showing that the Piccard iteration converges to a solution.
b) Suppose that there is a solution for $0 \le t \le T$. Show that $|x(t)| \le Mt + |x_0|$.
c) Deduce that there is a global solution for all $t \ge 0$.
d) What can go wrong if f is only continuous and bounded but not Lipschitz continuous. Give examples.

2. Let $x_0 = 1$ and $x_{n+1} = g(x_n)$, $n \ge 0$, where $g(x) = \dfrac{x^2+2}{2x}$. Show that $x_n \to \sqrt{2}$, $n \to \infty$.

3. The equations

$$\begin{cases} x' &= x(2 - x - y) \\ y' &= y(3 - 2x - y) \end{cases}$$

satisfy the conditions for competing species. Find the critical points and determine their nature. Explain why these equations make it mathematically possible, but extremely unlikely, for both species to survive.

4. Show that the following system has a closed orbit:

$$\begin{cases} x' &= 4x - y - x\, e^{x^2+y^2} \\ y' &= x + 3y - y\, e^{x^2+y^2} \end{cases}$$

5. Show that the planar system

$$\begin{cases} x' &= (1 - x^2 - y^2)x - y \\ y' &= x + (1 - x^2 - y^2)y \end{cases}$$

has a unique closed orbit γ and compute its Poincare map. Show that it is a periodic attractor.

6. Consider the equation

$$\ddot{x} + g(x)\dot{x} + f(x) = 0$$

where $g(x) \ge 0$, $f'(x) \ne 0$ when $f(x) = 0$. Describe the phase portrait when (a) $g(x) = 0$ for all x and (b) when $g(x) > 0$ for all x. Let $f(x) = x - x^3$.

7. Determine if the following systems are structurally stable in $D^2 = \{(x, y); x^2 + y^2 \le 1\}$:

$$(a) \quad \begin{cases} x' = -y \\ y' = x \end{cases} \qquad\qquad (b) \quad \begin{cases} x' = -x \\ y' = -y \end{cases}$$