Math 130B Solutions to Practice Midterm, Spring 2003, Lindblad.

1. (a) Let $\phi_0(t) = x_0$ and for $n \geq 0$ set $\phi_{n+1}(t) = x_0 + \int_0^t f(\phi_n(\tau))\, d\tau$. Then

$$|\phi_1(t) - \phi_0(t)| \leq C \int_0^t |f(\phi_0(\tau))|\, d\tau \leq Mt, \qquad \text{and for } n \geq 0:$$

$$|\phi_{n+1}(t) - \phi_n(t)| \leq \int_0^t \Big| f(\phi_n(\tau)) - f(\phi_{n-1}(\tau)) \Big|\, d\tau \leq \int_0^t L|\phi_n(\tau) - \phi_{n-1}(\tau)|\, d\tau$$

Let $e_n(t) = \sup_{0 \leq s \leq t} |\phi_{n+1}(s) - \phi_n(s)|$ and $E_n(t) = \sum_{k=0}^n e_n(t)$. Then

$$E_n(t) \leq L \int_0^t E_n(\tau)\, d\tau + Mt$$

Using Grönwalls lemma, i.e. let $U(t)$ denoted the right hand side. Then $U'(t) = LE_n(t) + M \leq LU(t) + M$. Multiplying by the integrating factor e^{-Lt} gives $(U(t)e^{-Lt})' \leq Me^{-Lt}$. Integrating both sides from 0 to t gives

$$U(t)e^{-Lt} \leq \int_0^t Me^{-L\tau}\, d\tau \leq \frac{M}{-L} e^{-L\tau}\Big|_0^t \leq \frac{M}{L}$$

and hence

$$E_n(t) \leq U(t) \leq ML^{-1}e^{Lt}$$

Since this is true for any n it follows that

$$\sum_{n=0}^{\infty} \sup_{0 \leq t \leq T} |\phi_{n+1}(t) - \phi_n(t)| \leq ML^{-1}e^{LT}$$

Given any fixed $\varepsilon > 0$ and $T < \infty$ there is an N such that

$$\sum_{n=N}^{\infty} \sup_{0 \leq t \leq T} |\phi_{n+1}(t) - \phi_n(t)| \leq \varepsilon$$

It therefore follows that

$$|\phi_n(t) - \phi_m(t)| = \Big| \sum_{k=m}^{n-1} \big(\phi_{k+1}(t) - \phi_k(t)\big) \Big| \leq \sum_{k=m}^{n-1} \big|\phi_{k+1}(t) - \phi_k(t)\big| \leq \varepsilon, \qquad \text{if} \quad n \geq m \geq N$$

i.e. $\{\phi_n\}$ is a Cauchy sequence in $C([0,T])$. Since this space is complete it follows that there a $\phi \in C([0,T])$, such that $\sup_{0 \leq t \leq T} |\phi_n(t) - \phi(t)| \to 0$ as $n \to \infty$. It therefore follows that the limit satisfies the integrated form of the equation:

$$\phi(t) = x_0 + \int_0^t f\big(\phi(\tau)\big)\, d\tau$$

(b) $|x(t)| = \Big| x_0 + \int_0^t f(x(\tau))\, d\tau \Big| \leq |x_0| + \int_0^t |f(x(\tau))|\, d\tau \leq |x_0| + \int_0^t M\, d\tau = |x_0| + Mt.$

(c) Let T be the largest time such that there is a solution for $0 \leq t < T$. We will

start by assuming that $T < \infty$ and show that this leads to a contradiction and hence $T = \infty$. Then by (b) $|x(t)| \le C < \infty$ and $|x'(t)| \le M$ for $0 \le t < T$. It follows that $x_T = \lim_{t\to T} x(t)$ exist and that $x \in C^1([0,T])$. We now use the local existence theorem again with initial data x_T starting from $t = T$ and this gives existence of another solution $\tilde{x}$ in some interval $(T-\varepsilon, T+\varepsilon)$, $\varepsilon > 0$. By uniqueness that solution $\tilde{x}$ has to agree with x in the interval $(T-\varepsilon, T]$. Therefore we can extend x to be equal to $\tilde{x}$ when $t \in [T, T+\varepsilon)$ and the extension $x \in C^1([0, T+\varepsilon))$ contradicting the maximality of T.
d) Uniqueness will go wrong. Take for example $f(x) = |x|^{1/2}$. Then one solution to $x' = f(x)$ with initial data $x(0) = 0$ is $x(t) = 0$. and another solution is $x(t) = t^2/4$. This does not quit satisfy the condition that $|f(x)| \le M$, but one can modify f so it is equal to $|x|^{1/2}$ when $|x| \le 1$ and equal to 1 when $|x| \ge 1$ and we still have nonuniqueness for small t.

2. Then $\sqrt{2}$ is a fixed point for $g(x)$; $g(\sqrt{2}) = \sqrt{2}$. We claim that it is a contraction of the set $W_0 = \{x; x \ge 1\}$:

$$|g(x) - g(y)| \le \frac{1}{2}|x-y|, \qquad \text{if} \quad x, y \ge 1 \tag{2.1}$$

and $g(x) \ge 1$ if $x \ge 1$. Therefore, by the above lemma, if we set $x_0 = 1$ and $x_{n+1} = g(x_n)$, for $n \ge 0$ then $x_n \to \sqrt{2}$, as $n \to \infty$. In fact,

$$x_0 = 1, \quad x_1 = 1.5, \quad x_2 = 1.41667..., \quad x_3 = 1.41422..., \cdots$$

To prove (2.1) we note that $|g'(s)| = |1/2 - 1/s^2| \le 1/2$, if $|s| \ge 1$ and hence

$$|g(x) - g(y)| = \Big| \int_y^x g'(s)\, ds \Big| \le \int_y^x |g'(s)| ds \le \frac{|x-y|}{2}, \quad \text{if} \quad x \ge y \ge 1.$$

3. Let $F(x,y) = \begin{bmatrix} x(2-x-y) \\ y(3-2x-y) \end{bmatrix}$. Then $DF(x,y) = \begin{bmatrix} 2-2x-y & -x \\ -2y & 3-2x-2y \end{bmatrix}$.
The critical points $F(x,y) = 0$: are $(0,0)$, $(1,1)$, $(0,3)$ and $(2,0)$.
Since $DF(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ has the eigenvalues $2>0$ and $3>0$, $(0,0)$ a source.
Since $DF(1,1) = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$ has eigenvalues $\pm\sqrt{2} - 1$, $(1,1)$ is a saddle.
Since $DF(0,3) = \begin{bmatrix} -1 & 0 \\ -6 & -3 \end{bmatrix}$ has eigenvalues -1 and -3, $(0,3)$ is a sink.
Since $DF(2,0) = \begin{bmatrix} -2 & -2 \\ 0 & -1 \end{bmatrix}$ has eigenvalues -2 and -1, $(2,0)$ is a sink.

Hence, there are no stable equilibrium points with both x and y nonvanishing. There are no closed orbits either. In fact, a closed orbit must be contained in a basic region, i.e. a region inbetween the curves where $x(2-x-y) = 0$ and $y(3-2x-y) = 0$. This is impossible since $x(t)$ and $y(t)$ are monotone in any basic region. Hence, by Poincare-Bendixson, the limit set is only the equilibrium points. Every trajectory will therefore tend to one of the equilibrium points. However, there is only a curve of initial data through the equilibrium $(1,1)$ that tends to $(1,1)$ all other initial conditions tend to one of the other equilibrium points. (12.3)

4. Let $r = \sqrt{x^2+y^2}$. Then

$$d(r^2)/dt = 2xx' + 2yy' = \ldots = 8x^2 + 6y^2 - 2(x^2+y^2)e^{x^2+y^2}$$

When $r = 1$ then $\frac{d}{dt}r^2 \geq 6 - 2e > 0$ and when $r = 2$ then $\frac{d}{dt}r^2 \leq 32 - 8e^4 < 0$. It follows that the annuls $K = \{(x,y);\ 1 \leq x^2+y^2 \leq 2^2\}$ is positively invariant. Hence by Theorem 3 there is either a closed orbit or an equilibrium point in K. However, there is no equilibrium in K, since $4x - y - x\,e^{x^2+y^2} = x + 3y - y\,e^{x^2+y^2}$ is equivalent to $(4x-y)/x = e^{x^2+y^2} = (x+3y)/y$ so $(4x-y)y = (x+3y)x$ or $x^2+y^2-xy = 0$ or $(x-y/2)^2 + 3y^2/4 = 0$. Hence the only equilibrium is $x=y=0$.

5. In polar coordinates the system take the form $r' = r(1-r^2)$ and $\theta' = 1$. $r = 1$ is a closed orbit. If $r > 1$ then $r' < 0$ so the system spirals towards smaller r and if $r < 1$ then $r' > 0$ so the system spirals towards larger r. Hence there can not be any closed orbit contained in either of the sets $r > 1$ or $r < 1$ and any orbit that intersects the curve $r = 1$ must coincide with this curve. Therefore the only closed orbit is $r = 1$.

The system can be solved by separation of variables $\frac{dr}{r(1-r^2)} = dt$ and partial fractions $\frac{1}{r(1-r^2)} = \frac{1}{r} + \frac{1}{1-r} - \frac{1}{1+r}$ which gives $\ln\frac{r}{1-r^2} = t + C$ so $\frac{r}{1-r^2} = Ce^t$ which after some work gives

$$r = \left(1 + (1/r_0^2 - 1)e^{-2t}\right)^{-1/2}, \qquad \theta = t + \theta_0$$

When t goes from 0 to 2π we have gone around one turn so the Poincare map is

$$P(r_0) = \left(1 + (1/r_0^2 - 1)e^{-4\pi}\right)^{-1/2}$$

and

$$P'(r_0) = e^{-4\pi}r_0^{-3}\left(1 + (1/r_0^2 - 1)e^{-4\pi}\right)^{-3/2}$$

Since $P'(1) < 0$ it follows from this that the orbit is a periodic attractor.

6. The equation is equivalent to the system $\begin{cases} x' = v \\ v' = -f(x) - g(x)v \end{cases}$

If $g(x) = 0$ it is Hamiltonian: $\begin{cases} x' = H_v(x,v) \\ y' = -H_x(x,v) \end{cases}$, where $H(x,v) = \dfrac{v^2}{2} + F(x)$ and $F(x) = \displaystyle\int_0^x f(s)\,ds$. Then $H(x,v)$ is constant along orbits.

In general $\dfrac{d}{dt}H(x,v) = -g(x)v^2$. In case $g(x) = 0$ for all x the solution curves has can be closed orbits or trajectories that escape to infinity. In case $g(x) > 0$ for all x the energy is decaying along orbits so there can be no closed orbits.

Let us describe the particular case when $f(x) = x - x^3$. The derivative is $\begin{bmatrix} 0 & 1 \\ -f'(x) - g'(x)v & -g(x) \end{bmatrix}$. The critical points are at $v = 0$ and $x = 0$ or $x =$

± 1. Then $\begin{bmatrix} 0 & 1 \\ -f'(x) & -g(x) \end{bmatrix}$. The eigenvalues are $\lambda^2 + g(x)\lambda + f'(x) = 0$ so $(\lambda + g(x)/2)^2 = g(x)^2/4 - f'(x)$ and hence

$$\lambda = -g(x)/2 \pm \sqrt{g(x)^2/4 - f'(x)}$$

If $x = 0$ and $g(x) = 0$ then $\lambda = \pm i$ so it is a center. and if $x = 0$ and $g(x) > 0$ then the real part of both eigenvalues are negative so it is a sink. If $x = \pm 1$ and $g(\pm 1) = a > 0$ then $\lambda = a + \pm\sqrt{2 + a^2/4}$ so it is a saddle.

If $g(x) = 0$ then $H(x, v) = c$ is constant along orbits. Since $F(x) = x^2/2 - x^4/4$ $H(x, v) = v^2/2 + x^2/2 - x^4/4 = c$. Completing the square gives:

$$v^2 = (x^2 - 1)^2/2 + (4c - 1)/2$$

If $c > 1/4$ then the solution curves are given by

$$v = \sqrt{(x^2 - 1)^2/2 + (4c - 1)/2}, \quad \text{or} \quad v = -\sqrt{(x^2 - 1)^2/2 + (4c - 1)/2}$$

If $c = 1/4$ the solution curves are

$$v = (x^2 - 1)/\sqrt{2}, \quad \text{or} \quad v = (1 - x^2)/\sqrt{2}$$

If $0 < c < 1/4$ there are two sorts of solution curves. The closed orbits contained in the set $-1 < x < 1$:

$$v^2 + x^2(1 - x^2/2) = 2c$$

(After a nonlinear stretching of the x axis $z = x\sqrt{1 - x^2/2}$, these are circles.) and trajectories escaping to infinity contained in either of the sets $x > 1$ or $x < -1$:

$$x^2 - 1 = \sqrt{2v^2 + (1 - 4c)},$$

or

$$x = \pm\sqrt{1 + \sqrt{2v^2 + (1 - 4c)}}$$

For $c < 0$ there are only the trajectories in $x > 1$ or $x < 1$.

It might be a good idea to use matlab or mathematica to do a counter plot of the level sets of $H(x, v)$.

Alternatively one can write it as the system $\begin{cases} x' = y - G(x) \\ y' = -f(x) \end{cases}$, where $G(x) = \int_0^x g(s)\, ds$.
If $H(x, y) = y^2/2 + F(x)$ it then follows that $\frac{d}{dt}H(x, y) = -f(x)G(x)$.

7. (a) is not structurally stable. In fact consider the perturbed system

$$\begin{cases} x' = -y + \mu x \\ y' = x + \mu y \end{cases}$$

The difference between the right hand sides when $\mu = 0$ or when μ is small is bounded by $C|\mu|(|x| + |y|)$ so on any compact set K we can make the perturbation as small as possible. The eigenvalues are

$$\lambda = \mu \pm i$$

so if $\mu = 0$ the origin is a center and the solution curves are circles. If $\mu < 0$ it is a sink and the solution curves are spirals towards the origin. If $\mu > 0$ it is a source and the solution curves are spirals out towards infinity. Clearly these case are not topologically equivalent, i.e. there can be no map taking the plane to the plane which maps the solution curves of the different pictures to each other. $\mu = 0$ is called a bifurcation point.

(b) is structurally stable close to the origin. This follows from applying Theorem 16.3.1.