Appendix: The Inverse and Implicit Function Theorems.

Theorem 1. *Suppose that $f : \mathbf{R}^n \to \mathbf{R}^n$ is C^1. Let $f(x_0) = y_0$ and suppose that*

$$Df(x_0) = \frac{\partial f}{\partial x}(x_0) \tag{11.2.3}$$

is invertible. Then for y close to y_0 there is an unique x close to x_0 such that

$$f(x) = y \tag{11.2.4}$$

Furthermore $x = x(y)$ is a C^1 function of y close to y_0.

Recall that a function of one variable $y = f(x)$ is invertible if $f'(x) \neq 0$. For a function of several variables we have by Taylor's formula

$$y - y_0 = f(x) - f(x_0) = (Df)(x_0)(x - x_0) + O\big(|x - x_0|^2\big) \tag{11.2.5}$$

where the derivative $Df(x_0) : \mathbf{R}^n \to \mathbf{R}^n$ is the linear map that best approximates the function close to a point and $O\big(|x - x_0|^2\big)$ means terms that are bounded by a constant times $|x - x_0|^2$ and hence small when $|x - x_0|$ is small. Therefore, as a first approximation we must be able to invert the linear linear map, and we get that $x - x_0 = (Df(x_0))^{-1}(y - y_0) + O(|y - y_0|^2)$.

The proof of Theorem 1 uses the contraction mapping theorem. First by a translation replacing $f(x)$ by $f(x+x_0)-y_0$ we can reduce to the case when $x_0 = y_0 = 0$. Furthermore by by multiplying both sides of (11.2.6) by the matrix $(Df(x_0))^{-1}$ and making a change of variables replacing y by $(Df(x_0))^{-1}y$ we may assume that the equation (11.2.4) takes the form

$$y = x + \phi(x) \tag{11.2.6}$$

where $\phi(x)$ is small; $\phi(0) = 0$ and $D\phi(0) = 0$. We seek a solution in the form

$$x = y + \psi(y) \tag{11.2.7}$$

Then for $\phi(y)$ we obtain the equation $\psi(y) = -\phi\big(y + \psi(y)\big)$ Consequently, the function ψ being sought is a fixed point of the mapping T defined by the formula

$$(T\psi)(y) = -\phi\big(y + \psi(y)\big) \tag{11.2.8}$$

Problem 1: Show that T is a contraction in some norm for y sufficiently small and use this to prove Theorem 1. You have to use that since ϕ is continuously differentiable and $D\phi(0)=0$ there is a neighborhood $|y|<\delta$, such that $\|D\phi(y)\|<\varepsilon$.

Theorem 2. *Suppose that $F : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n$ is C^1. Let $F(x_0, y_0) = c_0$ and suppose that*

$$\frac{\partial F}{\partial y}(x_0, y_0) \tag{11.2.9}$$

is invertible. Then for x close to x_0 there is a unique $y = g(x)$ close to y_0 such that

$$F(x, g(x)) = 0 \tag{11.2.10}$$

Furthermore $y = g(x)$ is a C^1 function of x close to y.

Problem 2: Show that Theorem 2 follows from Theorem 1, by considering the mapping $f(x, y) = \big(x, F(x, y)\big)$.

Problem 3: Suppose that $F(x, y, z)$ is a function of 3 variables, $F(x_0, y_0, z_0) = 0$, and $\operatorname{grad} F(x_0, y_0, z_0) \neq 0$. Use Theorem 2 to deduce that close to (x_0, y_0, z_0) the equation $F(x, y, z) = c_0$ is a surface, i.e. show that one of the variables say z (if $\partial F/\partial z \neq 0$ can by written as a graph $z = g(x, y)$ so that $F(x, y, g(x, y)) = 0$.