Solutions to Homework 9

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Problem 14.9: Show that the system

$$\begin{aligned}\dot{x} &= 10y - 10x \\ \dot{y} &= 28x - y + xz \\ \dot{z} &= xy - \frac{8}{3}z,\end{aligned}$$

is not chaotic for $x, y, z > 0$.

Solution: We begin by finding the equilibrium points. A point (x, y, z) is an equilibrium point if it satisfies

$$\begin{aligned}10y - 10x &= 0 \\ 28x - y + xz &= 0 \\ xy - \frac{8}{3}z &= 0.\end{aligned}$$

This means that we must have $x = y$ and, therefore, $x(27 + z) = 0$. Thus we have either $x = 0$ or $z = -27$. Since we are to suppose that $x, y, z > 0$ we must therefore have $x = 0$. Thus $x = y = 0$ and $xy - \frac{8}{3}z = 0$ implies that $z = 0$. Thus the only equilibrium point is the origin. Now we linearise around this equilibrium point and obtain the associated linearised system

$$\dot{\begin{pmatrix} u \\ v \\ w \end{pmatrix}} = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

which has eigenvalues $-\frac{8}{3}$ and

$$\frac{-11 \pm \sqrt{121 + 4 \times 270}}{2},$$

which means that the origin is hyperbolic and a saddle. Let $L_1(x, y, z) = xyz$. Then L_1 is positive for $x, y, z > 0$ and

$$\begin{aligned}\dot{L}_1(x, y, z) &= \dot{x}yz + x\dot{y}z + xy\dot{z} \\ &= (10y - 10x)yz + x(28x - y + xz)z + xy(xy - \frac{8}{3}z) \\ &= 10y^2z - 10xyz + 28x^2z - xyz + x^2z^2 + x^2y^2 - \frac{8}{3}xyz \\ &= (-10 - 1 - \frac{8}{3})xyz + 10y^2z + 28x^2z + x^2z^2 + x^2y^2.\end{aligned}$$

Suppose that (x, y, z) is a point in the set $\{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1, \text{ and } x, y, z > 0\}$. Then the point (tx, ty, tz), for t a real number, is a point on a line through the origin and (x, y, z) and

$$\dot{L}_1(tx, ty, tz) = t^3 \left((-10 - 1 - \frac{8}{3})xyz + 10y^2z + 28x^2z + t(x^2z^2 + x^2y^2) \right).$$

Since $x, y, z > 0$ there is $T(x, y, z)$ such that for $t > T(x, y, z)$ we have $\dot{L}_1 > 0$. Another way to see this is to use $L_2(x, y, z) = \frac{ax^2+by^2+cz^2}{2}$, where a, b, c are to be determined.

$$\begin{aligned}\dot{L}_2(x, y, z) &= ax\dot{x} + by\dot{y} + cz\dot{z} \\ &= ax(10y - 10x) + by(28x - y + xz) + cz(xy - \frac{8}{3}z) \\ &= xy\left((10a + 28b) + (b + c)z\right) - (10ax^2 + by^2 + \frac{8c}{3}z^2).\end{aligned}$$

If we pick $a = \frac{1}{10}$, $b = 1$, and $c = \frac{3}{8}$ then last term cleans up a little. Then we have

$$\dot{L}_2(x, y, z) = xy\left(29 + \frac{11}{3}z\right) - (x^2 + y^2 + z^2).$$

Again, suppose that (x, y, z) is a point in $\{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1, \text{ and } x, y, z > 0\}$. Then

$$\dot{L}_2(tx, ty, tz) = t^2\left(xy\left(29 + \frac{11}{3}zt\right) - (x^2 + y^2 + z^2)\right).$$

Again, there is $T(x, y, z)$ such that for $t > t(x, y, z)$ we have $\dot{L}_1 > 0$. Either way, for initial conditions (tx, ty, tz) with $t > T(x, y, z)$ the trajectory continues outward. For initial conditions (tx, ty, tz) with $t < T(x, y, z)$, the trajectory goes towards the origin. So no chaos here $-$ the flow is entirely predictable.

Extra problem: The following system is a model for the Earth's magnetic field:

$$\begin{aligned} \dot{x} &= -bx + yz \\ \dot{y} &= -by + xz - ax \\ \dot{z} &= 1 - xy, \end{aligned}$$

where a and b are positive constants.

(a) Find the equilibria.

(b) Show that volume decreases along the flow.

(c) Try to determine the types of the equilibria.

Solution:

(a) A point (x, y, z) is an equilibrium point if it satisfies

$$\begin{aligned} -bx + yz &= 0 \\ -by + xz - ax &= 0 \\ 1 - xy &= 0. \end{aligned}$$

It is clear from the last equation that x and y cannot be 0. The last equation also implies that $y = \frac{1}{x}$. Thus the first equation implies that $bx = \frac{z}{x}$, so $x^2 = \frac{z}{b}$. The middle equation looks like this

$$-\frac{b}{x} + x(z - a) = 0,$$

which implies

$$\begin{aligned} b &= x^2(z - a) \\ &= \frac{z}{b}(z - a). \end{aligned}$$

Thus $0 = z^2 - az - b^2$, which has solution

$$z = \frac{a \pm \sqrt{a^2 + 4b^2}}{2}.$$

Note that $b > 0$ and therefore

$$\frac{a - \sqrt{a^2 + 4b^2}}{2} < 0.$$

Thus this cannot be the z coordinate of an equilibrium point, since we would have

$$x = \sqrt{\frac{a - \sqrt{a^2 + 4b^2}}{2b}}$$

and

$$y = \frac{1}{\sqrt{\frac{a - \sqrt{a^2+4b^2}}{2b}}}.$$

Thus there are only two equilbrium points and they are

$$(x_0, y_0, z_0) = \left(-\sqrt{\frac{a + \sqrt{a^2 + 4b^2}}{2b}}, \frac{1}{-\sqrt{\frac{a+\sqrt{a^2+4b^2}}{2b}}}, \frac{a + \sqrt{a^2 + 4b^2}}{2} \right),$$

and

$$(x_1, y_1, z_1) = \left(\sqrt{\frac{a + \sqrt{a^2 + 4b^2}}{2b}}, \frac{1}{\sqrt{\frac{a+\sqrt{a^2+4b^2}}{2b}}}, \frac{a + \sqrt{a^2 + 4b^2}}{2} \right),$$

(b) Define

$$F(x, y, z) = \begin{pmatrix} -bx + yz \\ -by + xz - ax \\ 1 - xy \end{pmatrix}.$$

Then divergence of this vector-field is

$$\begin{aligned} (\mathrm{div}F)(x, y, z) &= -b - b + 0 \\ &= -2b, \end{aligned}$$

which means that along the flow generated by F, the volume element is shrinking.

(c) To determine what type of equilibrium point (x_0, y_0, z_0) is we linearise around (x_0, y_0, z_0) and obtain the associated linearised system

$$\dot{\begin{pmatrix} u \\ v \\ w \end{pmatrix}} = \begin{pmatrix} -b & z_0 & y_0 \\ z_0 - a & -b & x_0 \\ -y_0 & -x_0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

The characteristic polynomial of the above matrix is

$$\begin{aligned} p(x) &= -x^3 - 2bx^2 + x\left(-\frac{2b}{a + \sqrt{a^2 + 4b^2}} - \frac{a\sqrt{a^2 + 4b^2}}{2b} - \frac{\sqrt{a^2 + 4b^2}}{2b} + \frac{a\sqrt{a^2 + 4b^2}}{2b^2} - b^2 - \frac{a}{2b} + \frac{a^2}{2b^2} + \frac{a^2}{2b} + 1 \right) \\ &+ \left(\frac{a}{2} - \frac{a}{b} - \frac{\sqrt{a^2 + 4b^2}}{2} - \frac{\sqrt{a^2 + 4b^2}}{b} - \frac{2b^2}{a + \sqrt{a^2 + 4b^2}} \right) \\ &= -x^3 - 2bx^2 + x\frac{\sqrt{a^2 + 4b^2}\left((a - b)^2 + a^2 + b^2 + 2b^4\right) + 2a^3 + 6ab^2 - 2a^2b - 4ab^3 - 2ab^4 - 8b^3}{2b^2(a + \sqrt{a^2 + 4b^2})} \\ &+ \frac{-4a\sqrt{a^2 + 4b^2} - 4a^2 - 4b^3 - 8b^2}{2b(a + \sqrt{a^2 + 4b^2})}, \end{aligned}$$

and for matrix for the equilibrium point (x_1, y_1, z_1) the characteristic polynomial is the same. Since a and b are positive we can write $a = mb$, for some $m \neq 0$. From this we see that for large enough b the coefficient of x in the above polynomial is negative. Thus for such b − and remember $a = mb$ so b determines a − all the eigenvalues are positive. Thus for such b both of the equlibrium points are sources.