Math 130B Practice Final, Spring 2009, Lindblad.

1. Consider the system

$$x' = x^2 + y, \qquad y' = x - y + a$$

(a) Explain how the nullclines change as a increases from 0 to positive values, and explain why a bifurcation is expected for some positive values of a.
(b) Sketch the phase-portrait of the system when a is greater than the bifurcation value.
(c) Determine the a for which there is bifurcation.

2. Consider the system

$$x' = -y - x(2 - x^2 - y^2), \qquad y' = x$$

(a) Use a Lyapunov function of the form $L(x, y) = ax^2 + by^2$, where $a, b > 0$ to investigate the stability of the equilibrium point that the origin. If the origin is asymptotically stable, what can you say about the size of its basin of attraction?
(b) What happens to trajectories which do not go to the origin, as $t \to \infty$?

3. Consider the planar system in polar coordinates:

$$r' = r(2 - r), \qquad \theta' = r(1 - \cos\theta)$$

(a) Show that there are exactly two equilibrium points, one of which is the origin. Show that the origin is a source (the other is actually not hyperbolic).
(b) Show that the ray $\theta = 7r/2$, as well as the circle $r = 2$, are invariant sets.
(c) Explain why the circle r = 2 is not a periodic solution.
(d) Show that the w-limit set of every point (other than the origin) is the second equilibrium point.
(e) Recalling the precise definition of stability, explain why nevertheless this second equilibrium point is not stable!

4. This problem concerns the Hamiltonian system of equations whose Hamiltonian function is $H(x, y) = (x - 2)[(x + 1)^2 - 4y^2]$.
(a) Write the differential equations of this Hamiltonian system.
(b) Find all equilibrium points. Verify that H has the same value at three of them.
(c) The three equilibrium points at which H has the same value are the vertices of a triangle T. Show that H has the same constant value on the sides of T also. What kind of equilibria are the vertices of T?
(d) Show that all nonconstant trajectories starting in the interior of T remain there and are periodic.

5. For the system

$$x' = x(y-1), \qquad y' = x + y - 2y^2,$$

(a) Find the equilibria and classify them.
(b) Show that there are no closed orbits.
(c) Sketch the phase portrait.

6. Consider the Lienard system

$$x' = \mu(y - f(x)), \qquad y' = -x, \qquad \text{where} \quad f(x) = x(|x|-1).$$

It has a unique limit cycle, depending on μ that intersects the positive y axis at y_μ.
(a) Sketch the solutions in the phase plane when $\mu = 1$ starting at large $(0, y_\mu)$ for one small value $y < y_\mu$ and one large value of $y > y_\mu$.
(b) Sketch the limit cycle in the phase plane for very very large μ.

7. The equations

$$\begin{cases} x' &= x(2 - x - y) \\ y' &= y(3 - 2x - y) \end{cases}$$

satisfy the conditions for competing species.
(a) Find the critical points and determine their nature.
(b) Explain why these equations make it mathematically possible, but extremely unlikely, for both species to survive.

8. The equations of motions in the Newtonian Central force system are given by

$$\mathbf{X}'' = \mathbf{V}, \qquad \mathbf{V}' = -\,\mathbf{grad}\, U, \qquad \text{where} \quad U = -\frac{1}{r}, \quad r = |\mathbf{X}|.$$

a) Show that the energy $E = |\mathbf{V}|^2/2 + U(\mathbf{x})$ is conserved; $E' = 0$ along a trajectory.
b) Show that the angular momentum $\mathbf{X} \times \mathbf{V}$ is conserved.