Solutions to Practice Midterm

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Problem 1: Consider the system

$$\begin{aligned} x' &= x^2 + y \\ y' &= x - y + a \end{aligned}$$

1. Explain how the nullclines change as a increases from 0 to positive values, and explain why a bifurcation is expected for some positive values of a.
2. Sketch the phase-portrait of the system when a is greater than the bifurcation value.
3. Determine the a for which there is bifurcation.

Solution: The first two questions are best answered using drawings so they are on the pdf with the sketches in it. To find the value of a for which there is bifurcation we need to find the value of a for which the curve $\{(x, y) \in \mathbf{R}^2 : y = -x^2\}$ and the curve $\{(x, y) \in \mathbf{R}^2 : y = x + a\}$ meet exactly once. This occurs at a point (x, y) where the line $\{(x, y) \in \mathbf{R}^2 : y = x + a\}$ has the same gradient as $\{(x, y) \in \mathbf{R}^2 : y = -x^2\}$. That is, at $(-\frac{1}{2}, -\frac{1}{4})$. Now we need to find the a for which the line $\{(x, y) \in \mathbf{R}^2 : y = x + a\}$ passes through $(-\frac{1}{2}, -\frac{1}{4})$. This happens when $a = \frac{1}{4}$.

Problem 2: Consider the system

$$\begin{aligned} x' &= -y - x(2 - x^2 - y^2) \\ y' &= x \end{aligned}$$

1. Use a Lyapunov function of the form

   $$L(x, y) = ax^2 + by^2,$$

   where $a, b > 0$ to investigate the stability of the equilibrium point that the origin. If the origin is asymptotically stable, what can you say about the size of its basin of attraction?
2. What happens to trajectories which do not go to the origin, as $t \to \infty$?

Solution:

1. To show that the origin is an asymptotically stable equilibrium point, we must show that L, defined as above, is a strict Lyapunov function in some neighborhood of $(0, 0)$. It is clear that $L(x, y) \geq 0$ for all (x, y) and also that $L(x, y) = 0$ if $(x, y) = (0, 0)$. It remains to check that $\dot{L}$ is strictly negative in a neighborhood containing the origin. We have

$$\begin{aligned} \dot{L}(x, y) &= \begin{pmatrix} 2ax \\ 2by \end{pmatrix} \cdot \begin{pmatrix} -y - x(2 - x^2 - y^2) \\ x \end{pmatrix} \\ &= 2xy(b - a) - 2ax^2(2 - x^2 - y^2). \end{aligned}$$

We can see from this that if we pick $a = b = 1$ then L is a Lyapunov function in the ball of radius $\sqrt{2}$ with center the origin, so the origin is stable. We can also see that $\dot{L}$ is strict on the set $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 2 \text{ and } x \neq 0\}$. At a point (x, y) with $x = 0$ and $y \neq 0$, we see that $x' = -y$ and $y' = 0$, so the trajectory will go into the set $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 2 \text{ and } x \neq 0\}$ again. Therefore, even though L is not a strict Lyapunov function on $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 2 \text{ and } x \neq 0\}$, the origin is still asymptotically stable, and $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 2\}$ is contained in its basin of attraction.

2. If we pick initial conditions (x_0, y_0) on the circle $C = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 2\}$. Then the trajectory will be

$$\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 2\}.$$

Thus, a solution which starts on the circle C remains there. For solutions which start in the set $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 > 2 \text{ and } x \neq 0\}$, we see that

$$\dot{L} = -2x^2(2 - x^2 - y^2) > 0,$$

so these solutions tend to infinity. If we have initial conditions with $x_0 = 0$ then we know that the trajectory will end up in the region $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 > 2 \text{ and } x \neq 0\}$ soon, so all solutions in $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 > 2\}$ tend to infinity.

Problem 3: Now consider the system

$$\begin{aligned} \dot{r} &= r(2 - r) \\ \dot{\theta} &= r(1 - \sin(\theta)). \end{aligned}$$

1. Show that there are exactly two equilibrium points, one of which is a source at the origin.

2. Show that $A = \{(r, \theta) \in [0, \infty) \times [0, 2\pi) : r = 2\}$ and $B = \{(r, \theta) \in [0, \infty) \times [0, 2\pi) : \theta = \frac{\pi}{2}\}$ are invariant sets.

3. Explain why B is not periodic.

4. Show that for $p \neq (0, 0)$ we have $\omega(p) = \{(r, \frac{\pi}{2})\}$

5. Show that the equilibrium point at $(r, \frac{\pi}{2})$ is nevertheless, not stable.

Solution:

1. For $r = 0$ we certainly have $\dot{r} = 0$ and $\dot{\theta} = 0$. Now suppose that we have $r > 0$ and $\dot{r} = 0$ and $\dot{\theta} = 0$. Then we have $r = 2$, and $\sin(\theta) = 1$, that is $\theta = \frac{\pi}{2}$. Thus there are two equilibrium points, one at the origin an one at $(2, \frac{\pi}{2})$. Please note that these are the polar coordinates of the points. We see that for r with $0 < r < 2$, $\dot{r} > 0$ so the origin is a source.

2. For solutions starting on A we have $\dot{r} = 0$, so A is invariant. For solutions starting on B we have $\theta = \frac{\pi}{2}$, which means that $\dot{\theta} = 0$, so B is invariant.

3. The circle A is not a periodic solution because it contains an equilibrium point.

4. This is best explained by a picture.

5. For $(2, \frac{\pi}{2})$ to be stable we require that for all neighborhoods U of $(2, \frac{\pi}{2})$, there is a neighborhood V of $(2, \frac{\pi}{2})$, such that if we start a solution inside V it remains inside U. This is not the case, if we, for instance, start a solution at the right place on A, because it goes away from the equilibrium point for a while.