Solutions to Practice Midterm

By Håkan Nordgren

Problem 1: Consider the non-linear system

$$\begin{aligned} x' &= x^2 + y \\ y' &= x - y \end{aligned}$$

1. Locate the equilibrium points and use linearisation to determine their type.
2. Draw the nullclines and indicate the direction of the vector-field along them.
3. Indicate the direction of the vector-field in the basic regions between the nullclines.
4. Sketch a phase-portrait which is consistent with the information from above.

Solution:

1. A point (x, y) will be an equilibrium point if we have

$$\begin{aligned} x^2 + y &= 0 \\ x - y &= 0, \end{aligned}$$

which means that the points $(-1, -1)$ and $(0, 0)$ are the equilbrium points. The associated linearised system is

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 2x & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

So at the point $(-1, -1)$ the associated linearised system is

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

which has eigenvalues $\frac{-3-\sqrt{5}}{2}$ and $\frac{-3+\sqrt{5}}{2}$; that is, the equilibrium point is a sink. At the point $(0, 0)$ the associated linearised system is

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

which has eigenvalues $\frac{-1-\sqrt{5}}{2}$ and $\frac{-1+\sqrt{5}}{2}$; that is, the equilibrium point is a saddle.

2. The x-nullcline is the curve $\{(x, y) \in \mathbf{R}^2 : y = -x^2\}$. Along this nullcline, the vector-field points in the upward direction at points (x, y) with $0 < x - y$ and in the downward direction at points (x, y) with $0 > x - y$. Thus it points upward along the x-nullcline at points (x, y) with $x < -1$; downwards at points (x, y) with $-1 < x < 0$; and upwards

again at points (x, y) with $0 < x$. The y-nullcline is the curve $\{(x, y) \in \mathbf{R}^2 : y = x\}$. Along this nullcline, the vector-field points to the right at points (x, y) with $0 < x^2 + y$ and to the left at points (x, y) with $0 > x^2 + y$. Thus, along the y-nullcline, it points to the right at points (x, y) with $x < -1$; to the left at points (x, y) with $-1 < x < 0$; and to the right again at points (x, y) with $0 < x$.

Problem 2: Now consider the system

$$\begin{aligned} x' &= x^2 + y \\ y' &= x - y + a \end{aligned}$$

1. Explain how the nullclines change as a increases from 0 to positive values, and explain why a bifurcation is expected for some positive values of a.

2. Sketch the phase-portrait of the system when a is greater than the bifurcation value.

3. Determine the a for which there is bifurcation.

Solution: The first two questions are best answered using drawings so they are on the pdf with the sketches in it. To find the value of a for which there is bifurcation we need to find the value of a for which the curve $\{(x, y) \in \mathbf{R}^2 : y = -x^2\}$ and the curve $\{(x, y) \in \mathbf{R}^2 : y = x + a\}$ meet exactly once. This occurs at a point (x, y) where the line $\{(x, y) \in \mathbf{R}^2 : y = x + a\}$ has the same gradient as $\{(x, y) \in \mathbf{R}^2 : y = -x^2\}$. That is, at $(-\frac{1}{2}, -\frac{1}{4})$. Now we need to find the a for which the line $\{(x, y) \in \mathbf{R}^2 : y = x + a\}$ passes through $(-\frac{1}{2}, -\frac{1}{4})$. This happens when $a = \frac{1}{4}$.

Problem 3: Consider the system

$$\begin{aligned} x' &= -y - x(2 - x^2 - y^2) \\ y' &= x \end{aligned}$$

1. Use a Lyapunov function of the form

$$L(x, y) = ax^2 + by^2,$$

   where $a, b > 0$ to investigate the stability of the equilibrium point that the origin. If the origin is asymptotically stable, what can you say about the size of its basin of attraction?

2. What happens to trajectories which do not go to the origin, as $t \to \infty$?

Solution:

1. To show that the origin is an asymptotically stable equilibrium point, we must show that L, defined as above, is a strict Lyapunov function in some neighborhood of $(0,0)$. It is clear that $L(x,y) \geq 0$ for all (x,y) and also that $L(x,y) = 0$ if $(x,y) = (0,0)$. It remains to check that $\dot{L}$ is strictly negative in a neighborhood containing the origin. We have

$$\begin{aligned} \dot{L}(x,y) &= \begin{pmatrix} 2ax \\ 2by \end{pmatrix} \cdot \begin{pmatrix} -y - x(2 - x^2 - y^2) \\ x \end{pmatrix} \\ &= 2xy(b-a) - 2ax^2(2 - x^2 - y^2). \end{aligned}$$

   We can see from this that if we pick $a = b = 1$ then L is a Lyapunov function in the ball of radius $\sqrt{2}$ with center the origin, so the origin is stable. We can also see that $\dot{L}$ is strict on the set $\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 < 2 \text{ and } x \neq 0\}$. At a point (x,y) with $x = 0$ and $y \neq 0$, we see that $x' = -y$ and $y' = 0$, so the trajectory will go into the set $\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 < 2 \text{ and } x \neq 0\}$ again. Therefore, even though L is not a strict Lyapunov function on $\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 < 2 \text{ and } x \neq 0\}$, the origin is still asymptotically stable, and $\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 < 2\}$ is contained in its basin of attraction.

2. If we pick initial conditions (x_0, y_0) on the circle $C = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 = 2\}$. Then the trajectory will be

$$\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 = 2\}.$$

   Thus, a solution which starts on the circle C remains there. For solutions which start in the set $\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 > 2 \text{ and } x \neq 0\}$, we see that

$$\dot{L} = -2x^2(2 - x^2 - y^2) > 0,$$

   so these solutions tend to infinity. If we have initial conditions with $x_0 = 0$ then we know that the trajectory will end up in the region $\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 > 2 \text{ and } x \neq 0\}$ soon, so all solutions in $\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 > 2\}$ tend to infinity.

Problem 4: Consider the planar system

$$\begin{aligned} \dot{x} &= x - y - x^3 \\ \dot{y} &= x + y - y^3. \end{aligned}$$

1. Show that for sufficiently large a, the set $S_a = \{(x,y) \in \mathbf{R}^2 : -a \leq x \leq a, \text{ and } -a \leq y \leq a\}$ is positively invariant.

2. Assuming that the origin is the only equilibrium point, show that S_a contains a closed orbit.

Solution:

1. We need to show that if we make a large enough, the flow is not outward on the edges of S_a. Let us begin by considering the edge $E_a^1 = \{(x, y) \in \mathbf{R}^2 : x = -a, \text{ and } -a \leq y \leq a\}$. This edge has outward pointing unit normal $(-1, 0)$. Define

$$F(x, y) = \begin{pmatrix} x - y - x^3 \\ x + y - y^3 \end{pmatrix}.$$

Then $F(x, y) \cdot (-1, 0) = -x + y + x^3$. On the edge E_a^1, we have $x = -a$, and $-a \leq y \leq a$ so $F(x, y) \cdot (-1, 0) = a + y - a^3 \leq a + a - a^3 < 0$. for a large enough. Now consider the edge $E_a^2 = \{(x, y) \in \mathbf{R}^2 : y = a, \text{ and } -a \leq x \leq a\}$. This edge has outward pointing unit normal $(0, 1)$. Then $F(x, y) \cdot (0, 1) = x + y - y^3$. On the edge E_a^2, we have $y = a$, and $-a \leq x \leq a$ so $F(x, y) \cdot (0, 1) = x + a - a^3 \leq a + a - a^3 < 0.$, for $a > 0$ large enough. The edge $E_a^3 = \{(x, y) \in \mathbf{R}^2 : x = a, \text{ and } -a \leq y \leq a\}$ has outward pointing unit normal $(1, 0)$. Then $F(x, y) \cdot (1, 0) \leq a + a - a^3 < 0$, for $a > 0$ large enough. Finally, the edge $E_a^4 = \{(x, y) \in \mathbf{R}^2 : y = -a, \text{ and } -a \leq x \leq a\}$ has outward pointing unit normal $(0, -1)$. Then $F(x, y) \cdot (0, -1) \leq a + a - a^3 < 0$, for $a > 0$ large enough. Thus for a large enough, S_a is positively invariant.

2. We now linearize around the equilibrium point at the origin:

$$D(F)(x, y) = \begin{pmatrix} 1 - 3x^2 & -1 \\ 1 & 1 - 3y^2 \end{pmatrix}.$$

Thus

$$D(F)(0, 0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

which means that the origin is a source. Therefore we can find a neighborhood of the origin which is negatively invariant. Thus if we take the square S_a and delete this neighborhood, we have a positively invariant set which is compact, and contains no equilibrium points. Thus, by the Poincaré-Bendixson theorem, we know that for a point x in this set, $\omega(x)$ is a periodic orbit.