Math 150A Final Fall 98 Lindblad.

1. Let A, B and C be real numbers. Determine the second fundamental form at $(0,0,0)$ of the regular surface $z = Ax^2 + Bxy + Cy^2$ in $\mathbf{R}^3$, and its Gauss and mean curvature there. Also express the normal curvature along a direction $(x, y, 0) \in T_0 S$ as a function of A, B, C and the angle θ between $(1,0,0)$ and $(x, y, 0)$.

2. Consider the following surfaces $S_i \subset \mathbf{R}^3$:

$$S_1 = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 = 1\},$$
$$S_2 = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\},$$
$$S_3 = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 - z^2 = 1\},$$
$$S_4 = \{(x, y, z) \in \mathbf{R}^3 : x = 1\},$$
$$S_5 = \{(x, y, z) \in \mathbf{R}^3 : (z-1)^2 - y^2 - x^2 = 0\}.$$

Obviously, $p = (1,0,0)$ is a point of all of these surfaces. Decide which ones are locally isometric at p (that is, for which S_i, S_j does there exist an isometry $\phi : S_i \to S_j$ defined in a neighborhood U of p in S with $\phi(p) = p$?). Give reasons in each case. If you find two of these surfaces are isometric, give an explicit isometry.

3. Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function. Consider the plane curve $\gamma : \mathbf{R} \to \mathbf{R}^2, \gamma(t) = (t, f(t))$. Prove a formula for the curvature of γ at each point $\gamma(t_0)$. Use your result to determine the curvature of the parabolas $y = kx^2$ at the origin.

4. Consider the cylinder $S = \{(x, y, z) \in \mathbf{R}^3 : x^2 + z^2 = 1\}$. Introduce coordinates (θ, z) by $x = \cos(\theta), y = \sin(\theta), z = z$. For each integer n consider the map $\varphi_n : S \to S$ which in the above coordinates takes the form $(\theta, z) \mapsto (n\theta, z)$. Determine the differential $d(\varphi_n)_{(1,0,0)}$ of φ_n at $(1,0,0)$ and the image of the tangent vector $(1,0,1)$ at this point under $d(\varphi_n)_{(1,0,0)}$.

5. Consider the map $f : \mathbf{R}^2 \to \mathbf{R}^2 \setminus \{0\}$, $f(x, y) = (e^x \cos(y), e^x \sin(y))$. Show that f is conformal, when you consider $\mathbf{R}^2$ as the regular surface $z = 0$. Is f an isometry?

6. Assume that $\mathbf{x}(u, v)$ parameterize a regular surface with unit normal $\mathbf{N}$, Gaussian curvature K and mean curvature H. Assume furthermore that all the coordinate curves are principal. Consider the *parallel surface* $\mathbf{y}(u, v) = \mathbf{x}(u, v) + c\mathbf{N}(u, v)$ and assume that it is regular. Show that (a) $\mathbf{N}$ is a unit normal to y; (b) that $\mathbf{y}_u \times \mathbf{y}_v = (1 - 2Hc + Kc^2)\mathbf{x}_u \times \mathbf{x}_v$; and (c) use the interpretation of the Gaussian curvature in terms of areas and the Gauss map to show that the Gaussian curvature of $\mathbf{y}$ is $K/(1 - 2Hc + Kc^2)$.

7. Suppose α is a curve in $\mathbf{R}^3$ parameterized by arc-length with curvature $k \neq 0$ and torsion τ. Show that the curvature of $\beta = \alpha'$ is given by $\sqrt{1 + \frac{\tau^2}{k^2}}$.

8. Consider the Moebius strip given by rotating a stick of length 1 in the positive direction by 180 degrees while walking along a positively oriented circle of radius 2 in the x–y plane. Find a parametrization for this non-orientable surface. Define Gaussian curvature for non-orientable surfaces and compute it for this one.

9. Give examples for surfaces with constant Gauss curvature $K = -1$, $K = 0$, and $K = 1$. Determine all surfaces of revolution with constant Gauss curvature 0.

10. Let S be a regular surface. Prove that if a point $p \in S$ is elliptic, then there is a neighborhood U of p in S such that all the points of U are elliptic. Is the same true for hyperbolic, parabolic, or planar points? Give examples!