Lecture 1: 1.1 Introduction and overview.

Math 150A Differential Geometry and 150B Calculus on Manifolds can be seen as continuations of 20E Vector Calculus. In 150A we will learn about the Differential Geometry of Curves and Surfaces in space. The word geometry, comes from Greek Geo=earth and metria=measure. Geometry is the part of mathematics concerned with questions of size, shape and position of objects in space. Differential geometry uses the methods of differential and integral calculus to study the geometry. In 150B we will study Vector Calculus on Manifolds (which locally are hyper surfaces) and how it applies to physics. Let us now continue with briefly describing the content of 150A and how it related to 20E.

First we will see that a curve in space is determined by its initial point and direction and two scalars called the curvature and torsion at each point along the curve, that measures how fast the curve pulls away from the tangent line close to a point. What do you remember about curves from Vector Calculus? Arc length. Curvature. What is the curvature of a plane curve? At each point along a curve we are given the tangent line but we can also fit a circle that is tangent to the curve to second order and that best approximates the curve close to a point. This circle has a certain radius R and the inverse of the radius is called the curvature $\kappa = 1/R$. A plane curve is determined by giving the initial position and tangent line and the curvature everywhere along the curve. A space curve doesn't only go in a plane but also twists around and that is measured by the torsion. Locally it approximately lies in the plane containing the circle that best approximates it but that plan twists around.

Then we properly define what is meant by a regular surface, to be something that close to each point looks like a deformed plane. What is your picture of a surface from 20E? Something given as a level surface, e.g $z^2 = x^2 + y^2 + a^2$. What if $a = 0$ is this set a surface when $z = x = y = 0$? We call something a regular surface if things like this doesn't happen. I more useful description is something given as graph or as parameterized surface, e.g. the sphere $x^2 + y^2 + z^2 = a^2$ can be parameterized by spherical coordinates. It can also locally be written as graphs of the coordinate planes $z = \pm\sqrt{a^2 - x^2 - y^2}$. Note however, that you need several parameterizations to cover the whole surface. We show that one can do calculus for functions defined on surfaces.

We also introduce the first fundamental form which is used to measure lengths and areas on the surface itself. This is done as follows. We can measure lengths and areas in space and this gives us a measure of lengths and areas on curves and surfaces by measuring the lengths of tangent vectors.

Next we define the the second fundamental form and the principal curvatures of surface, i.e. the maximum and minimum curvatures of the curves of intersection of the surface with planes through the normal. The second fundamental form is the quadric form that best approximates a surface close to a point if the coordinates are chosen so that the linear part vanishes. These measure how fast the surface pulls away from the tangent plane in a neighborhood of a point on it.

We will show that a surface is determined by its first and second fundamental forms.

We will also prove Gauss Theorema Egregium, which states that the Gaussian curvature, i.e. the product of the principal curvatures, is invariant under isometries, i.e. maps that bend the surface without stretching it.

We will also study geodesics, which are the closest curves between points on a surface.

Gauss and Riemann asked how much of the geometry of a surface is independent of how it bends in space and can be described from creatures that live on the surface and can measure lengths on the surface but are unaware of space outside. This later lead to Einstein general theory of Relativity.