

## Lecture 2: 1.2 Parameterized Curves.

We say that a real valued function is (infinitely) differentiable or smooth if it has, at all points, derivatives of all orders (which are automatically continuous).

**Def** A *parameterized differentiable curve* is a map  $\alpha : I \rightarrow \mathbf{R}^3$  of an open interval  $I = (a, b)$  of the real line  $\mathbf{R}$  to three dimensional space  $\mathbf{R}^3$ .

Let  $\alpha(t) = (x(t), y(t), z(t))$  denote the point in space corresponding to parameter value  $t \in I$ .

That  $\alpha$  is differentiable means that each of the functions  $x(t), y(t), z(t)$  are differentiable.

$\alpha'(t) = (x'(t), y'(t), z'(t))$  is called the *tangent vector* (or the *velocity vector* if we think of the curve as the path of a particle)

The image set  $\alpha(I) \subset \mathbf{R}^3$  is called the *trace* of the curve  $\alpha$ .

**Ex** The two parameterized curves in the plane  $\alpha(t) = (\cos t, \sin t)$ ,  $t \in (0, 2\pi)$ , and  $\beta(t) = (\cos 2t, \sin 2t)$ , where  $t \in (0, \pi)$ , have the same trace, namely the unit circle. Note that the velocity vector of the first curve has the double length of the first.

**Ex** A parameterized curve in space is given by  $\alpha(t) = (\cos t, \sin t, t)$ . Its trace is a subset of the cylinder  $x^2 + y^2 = 1$ .

Let us start to review some vector calculus. The length of a vector  $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{R}^3$  is given by

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

The dot product of two vectors  $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{R}^3$  and  $\mathbf{v} = (v_1, v_2, v_3) \in \mathbf{R}^3$  are given by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}| |\mathbf{v}| \cos \theta,$$

where  $0 \leq \theta \leq \pi$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ .

The following formula for the derivative of a dot product will turn out to be useful:

$$\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t).$$

## 1.3 Arc Length.

**Def** A parameterized differentiable curve is said to be *regular* if  $\alpha'(t) \neq 0$  for all  $t \in I$ .

The arc length of a regular curve from a point  $t_0 \in I$  is by definition

$$s(t) = \int_{t_0}^t |\alpha'(t)| dt, \quad \text{where } |\alpha'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}.$$

is the length of the tangent vector  $\alpha'(t)$ . We have  $ds(t)/dt = |\alpha'(t)|$ . This says that the length along the curve is the integral of its speed along the curve.

It can happen that the parameter  $t$  is already the arc length in which case  $|\alpha'(t)| = 1$ . In fact, if this is not the case one can always make a change of parameter along a regular curve so it is parameterized by arc length. In fact since  $s'(t) > 0$  it is invertible with the inverse  $t(s)$ , which satisfies  $t'(s) = 1/s'(t)$ . The curve  $\beta(s) = \alpha(t(s))$  is the same curve parameterized by the arc length, in fact  $|\beta'(s)| = |\alpha'(t(s))|t'(s) = s'(t)t'(s) = 1$ .

**Ex** Find the arc length of the circle  $\alpha(t) = (\cos t, \sin t)$  and  $t \in (0, 2\pi)$ .

The *acceleration vector* of a curve is

$$\alpha''(t) = (x''(t), y''(t), z''(t))$$

Note that the curve  $\alpha$  is a constant speed straight line if and only if  $\alpha'' = 0$ . This follows from just integrating each coordinate function.

What direction does  $\alpha''(t)$  point in? If it always point in the direction of  $\alpha'(t)$  for and  $t$ , then in fact the trace of  $\alpha$  is just a line. The interesting thing is that if the arc length is the parameter, then  $\alpha''(t)$  is perpendicular to  $\alpha'(t)$ . This follows by differentiating  $|\alpha'(t)|^2 = \alpha'(t) \cdot \alpha'(t) = 1$ ;

$$0 = \frac{d}{dt}(\alpha'(t) \cdot \alpha'(t)) = 2\alpha'(t) \cdot \alpha''(t)$$

Note also that the shortest distance between two points is a line. If the points are  $\mathbf{p}$  and  $\mathbf{q}$  then the distance is  $|\mathbf{p} - \mathbf{q}|$  and if  $\mathbf{u} = (\mathbf{p} - \mathbf{q})/|\mathbf{p} - \mathbf{q}|$  then

$$\int_a^b \alpha'(t) \cdot \mathbf{u} dt = \dots = |\mathbf{p} - \mathbf{q}|$$

On the other hand

$$\int_a^b \alpha'(t) \cdot \mathbf{u} dt \leq \dots \leq \int_a^b |\alpha'(t)| dt$$

**1.4 The vector product in  $\mathbf{R}^3$ .** The definition of and notation for the vector product in the text is unusual. We define the vector product of  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$  to be the unique vector  $\mathbf{u} \wedge \mathbf{v}$  such that for all vectors  $\mathbf{w} = (w_1, w_2, w_3)$  we have

$$(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w} = \det(\mathbf{u}, \mathbf{v}, \mathbf{w}) \equiv \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$