(3) To show that the Frenet frames now coincide for all s, consider the function

$$\mathcal{D}(s) = T_\alpha(s) \cdot T_\beta(s) + N_\alpha(s) \cdot N_\beta(s) + B_\alpha(s) \cdot B_\beta(s)$$

which attempts to measure the deviation of the frames from being identical. Show that $\mathcal{D}(0) = 3$, $\mathcal{D}(s) \leq 3$ for all s and $\mathcal{D}(s_0) = 3$ only when the frames coincide at s_0.

(4) Show that $\mathcal{D}$ is constant by calculating $\mathcal{D}' = 0$. Use the Frenet equations. Hence, $\mathcal{D}(s) = 3$ for all s.

(5) Then, $\alpha'(s) = \beta'(s)$. Integrate and use initial conditions to show $\alpha(s) = \beta(s)$, hence showing that the original translation, rotation and reflection carried β into α.

1.6. Green's Theorem and the Isoperimetric Inequality

Closed curves hold a special place in the worlds of geometry and applications of mathematics. A curve $\alpha\colon [a,b] \to \mathbb{R}^2$ is *closed* if $\alpha(a) = \alpha(b)$ and $\alpha^{(n)}(a) = \alpha^{(n)}(b)$ for all $n > 0$, where $\alpha^{(n)}$ denotes the n-th derivative. In applications, closed curves often represent periodic orbits of physical systems. This periodicity then allows a complete understanding of the dynamical behavior of a system by simply analyzing what happens in a single period. In geometry, there is often an important connection between the closed curve and the region inside that it bounds. We will see this later, for instance, in the Gauss-Bonnet theorem. Here we want to present the most basic connection between a plane closed curve and the region inside; this connection is called the *isoperimetric inequality.* This inequality expresses a relation between the arclength of a simple closed curve in the plane and the area of the enclosed region. (Recall that a curve is *simple* if it has no self-intersections.)

Theorem 1.6.1 (The Isoperimetric Inequality). *Among all simple closed curves in the plane having a fixed length, the circle bounds the enclosed region of largest area. In particular, if C is a simple closed curve with length L enclosing an area A, then*

$$L^2 \geq 4\,\pi\,A$$

with equality holding only in the case of a circle.

The proof we give of the isoperimetric inequality is due to E. Schmidt, but we shall basically follow the approach of [**Che67**]. Also, compare the proof below with the calculus of variations approach in Example 7.6.9. Before we begin the proof, we need to recall a fundamental result of two-variable calculus.

Theorem 1.6.2 (Green's Theorem). *Let $P(x,y)$ and $Q(x,y)$ be two real valued (smooth) functions of two variables x and y defined on a simply connected region (i.e. a region without holes) $\mathcal{R}$ of the plane. Then*

$$\iint_{\mathcal{R}} \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\,dx\,dy = \int_C P\,dy - Q\,dx$$

where the right hand side is the (counterclockwise) line integral around the boundary C of the region $\mathcal{R}$.

Remark 1.6.3. Note that Green's theorem may be written in different forms. For instance, an equivalent form is given by

$$\iint_{\mathcal{R}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\, dy = \int_C P\, dx + Q\, dy.$$

In the rest of the book, we shall use whatever form of Green's theorem suits a given problem.

Example 1.6.4 (Green's Theorem and Area)**.** Let C be a closed curve enclosing a region $\mathcal{R}$. Take $P = x/2$ and $Q = y/2$. Green's theorem then gives

$$\text{Area}(\mathcal{R}) = \iint_{\mathcal{R}} dx\, dy = \iint_{\mathcal{R}} \frac{1}{2} + \frac{1}{2}\, dx\, dy = \frac{1}{2}\int_C x\, dy - y\, dx.$$

Therefore, the area inside a closed curve may be expressed as a line integral. Of course, the line integral is evaluated by parametrizing the curve as $\beta(t) = (x(t), y(t))$ for $a \le t \le b$ and taking $\frac{1}{2}\int_C x\, dy - y\, dx = \frac{1}{2}\int_a^b (x(t)\, y'(t) - y(t)\, x'(t))\, dt$. The same arguments say that there are also the following identifications: $\text{Area}(\mathcal{R}) = \int_C x\, dy = \int_C -y\, dx$.

Proof of Theorem 1.6.1. Let the curve C be parametrized by $\alpha(s) = (x(s), y(s))$ for $0 \le s \le L$ and assume, without loss of generality, that the parametrization is unit speed. Since C is a closed curve, we can place it between two parallel lines. We orient the picture so that these lines, m and n, are vertical and touch C at points $P = \alpha(0)$ and $Q = \alpha(s_0)$ (see Figure 1.23). We also place a comparison circle K inside the lines with radius r half the distance between the lines. We parametrize the circle by using the x-coordinate of α and making the y-coordinate obey the circle relation; $\beta(s) = (\beta_1(s), \beta_2(s))$, where $\beta_1(s) = x(s)$ and

$$\beta_2(s) = \begin{cases} -\sqrt{r^2 - x(s)^2} & 0 \le s \le s_0 \\ +\sqrt{r^2 - x(s)^2} & s_0 \le s \le L \end{cases}.$$

Note that $\beta(0) = \beta(L)$. For C, we can apply Green's theorem to compute the area inside to be $A_C = \int_C x\, dy = \int_0^L x(s)\, y'(s)\, ds$. For the circle, we have $\pi r^2 = A_K = \int_K -\beta_2(s)\,dx = \int_0^L -\beta_2(s)\,x'(s)\;ds$. Add these to get

$$\begin{aligned}
A_C + \pi r^2 &= \int_0^L xy' - \beta_2 x'\;ds \\
&\le \int_0^L |xy' - \beta_2 x'|\;ds \\
&= \int_0^L \sqrt{(xy' - \beta_2 x')^2}\;ds \\
&= \int_0^L \sqrt{x^2 y'^2 - 2xy'\beta_2 x' + {\beta_2}^2 x'^2}\;ds \\
&\le \int_0^L \sqrt{(x^2 + {\beta_2}^2)(x'^2 + y'^2)}\;ds \qquad \text{since } (xx' + \beta_2 y')^2 \ge 0 \\
&= \int_0^L \sqrt{x^2 + {\beta_2}^2}\;ds \qquad \text{since } \alpha \text{ has unit speed} \\
&= \int_0^L |\beta(s)|\;ds \\
&= \int_0^L r\;ds \qquad \text{since } \beta(s) = (x(s), \sqrt{r^2 - x(s)^2}) \\
&= r\,L.
\end{aligned}$$

The usual arithmetic-geometric-mean inequality implies that $\sqrt{A_C\,\pi\,r^2} \le (A_C + \pi\,r^2)/2$, so we obtain

$$\begin{aligned}
\sqrt{A_C\,\pi\,r^2} &\le \frac{r\,L}{2} \\
A_C\,\pi\,r^2 &\le \frac{r^2\,L^2}{4} \\
4\pi A_C &\le L^2
\end{aligned}$$

and this is the isoperimetric inequality. Of course, we must still consider the case of equality, so suppose that $4\pi A_C = L^2$. Then, by following backwards through the estimates above, we see that all of the inequalities are, in fact, equalities. In particular, the equality

$$\sqrt{A_C\,\pi\,r^2} = \frac{A_C + \pi\,r^2}{2} = \frac{r\,L}{2}$$

says that $A_C = \pi\,r^2$ and, consequently, $L = 2\,\pi\,r$. Furthermore, because the integrals above are equal and the respective integrands positive, we must also have

$$((x, \beta_2) \cdot (y', -x'))^2 = (xy' - \beta_2 x')^2 = r^2$$

where $\cdot$ denotes the dot product. But $|(x, \beta_2)| = r$, $|(y', -x')| = 1$ and $(x, \beta_2) \cdot (y', -x') = |(x, \beta_2)|\,|(y', -x')|\,\cos(\theta)$, so we must have $\cos(\theta) = 1$. Hence, $\theta = 0$ and (x, β_2) and $(y', -x')$ are in the same or opposite directions.

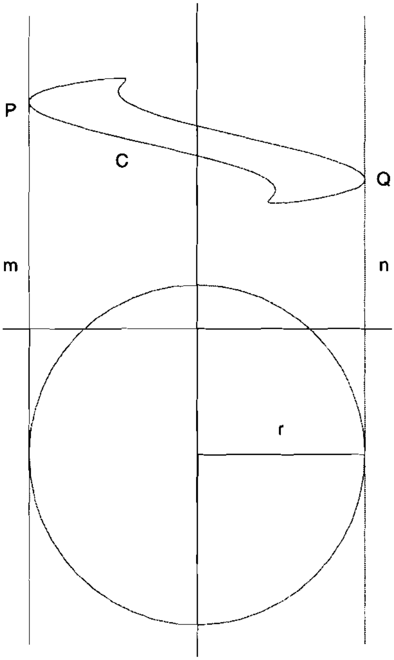

FIGURE 1.23. Closed curve and comparison circle

The lengths of the vectors then imply that $(x, \beta_2) = \pm r\,(y', -x')$. Therefore, $x = \pm r\,y'$ and $\beta_2 = \mp r\,x'$. Using $\beta_2 = \sqrt{r^2 - x^2}$, the second equation gives

$$\int \frac{1}{\sqrt{r^2 - x^2}}\,dx = \int \frac{1}{r}\,ds$$
$$\int d\theta = \frac{s}{r} + d$$

using $x = r\sin(\theta)$;

$$\theta = \frac{s}{r} + d$$
$$\arcsin\left(\frac{x}{r}\right) = \frac{s}{r} + d$$
$$x(s) = r\,\sin\left(\frac{s}{r} + d\right).$$

Now, $x'(s) = \cos(s/r + d)$ and $x'^2 + y'^2 = 1$, so we can solve to get $y(s) = \mp r\,\cos(s/r + d)$. Thus, up to signs, the parametrization for C is that of a circle of radius r;

$$\alpha(s) = \left(r\,\sin\left(\frac{s}{r} + d\right),\; r\,\cos\left(\frac{s}{r} + d\right)\right).$$

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