Lecture 9: 2.4 The differential of a map between surfaces.

We did the proof of Proposition 2.

We also talked about contractions, see next page below, in preparation for the proof of the inverse function theorem next time. (An alternative source of this is Rudin's analysis book used for 140.)

Appendix B: The inverse function theorem.

Contractions.
A map $T : W \to W$ is called a *contraction*, if for $x, y \in W$:

(B1) $$\|T(x) - T(y)\| \le K\|x - y\|, \qquad K < 1$$

A point $x \in W$ is called a *fixed point* if $T(x) = x$. We have:

Lemma 2. *Let $T : W_0 \to W_0$ be a contraction of a complete normed space W_0. Then T has a unique fixed point $x \in W_0$. In fact for any $x_0 \in W_0$, $x_k = T^k(x_0) = T \circ \cdots \circ T(x_0)$ (k times) converges to x; $\|x - x_k\| \to 0$, as $k \to \infty$.*

Proof. Using (B1) repeatedly we get

(B2) $$\|x_{k+1} - x_k\| = \|T(x_k) - T(x_{k-1})\| \le K\|x_k - x_{k-1}\| \le \cdots \le K^k\|x_1 - x_0\|$$

Here $\|x_1 - x_0\| = \|T(x_0) - x_0\| = C$ is a fixed constant. For $m > k$ we write $x_m - x_k = (x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + ... + (x_{k+1} - x_k)$ and estimate the norm of each term by (B2):

(B3) $$\|x_m - x_k\| \le \|x_m - x_{m-1}\| + \cdots + \|x_{k+1} - x_k\| \le (K^{m-1} + \cdots + K^{k-1})C$$

This is a geometric sum and since $K < 1$ the infinite sum converges; $\sum_{\ell=k-1}^{m-1} K^\ell \le \sum_{\ell=k-1}^{\infty} K^\ell = K^{k-1}\sum_{n=0}^{\infty} K^n = K^{k-1}/(1 - K)$. Hence

(B4) $$\|x_m - x_k\| \le \varepsilon(N) = \frac{CK^{N-1}}{1-K}, \qquad \text{if} \quad m, k \ge N,$$

where $\varepsilon(N) \to 0$ as $N \to \infty$, i.e. x_k is a Cauchy sequence.
The uniqueness follows from (B1); if $T(x) = x$ and $T(y) = y$ then $\|x - y\| = \|T(x) - T(y)\| \le K\|x - y\|$ and since $K < 1$ it follows that $\|x - y\| = 0$ so $x = y$. $\square$

Ex. Find an approximation for $\sqrt{2}$. Let

(B5) $$g(x) = \frac{x^2 + 2}{2x}$$

Then $\sqrt{2}$ is the only positive fixed point for $g(x)$; $g(\sqrt{2}) = \sqrt{2}$. We claim that $x \to g(x)$ is a contraction of the set $W_0 = \{x; x \ge 1\}$:

(B6) $$|g(x) - g(y)| \le \frac{1}{2}|x - y|, \quad \text{if} \quad x, y \ge 1, \qquad \text{and} \qquad g(x) \ge 1 \quad \text{if} \quad x \ge 1$$

By the above lemma, if we set $x_0 = 2$ and $x_{n+1} = g(x_n)$, for $n \ge 0$ then $x_n \to \sqrt{2}$, as $n \to \infty$:

(B7) $$x_0 = 2, \quad x_1 = 1.5, \quad x_2 = 1.41667..., \quad x_3 = 1.41422..., \cdots$$

To prove (B6) we note that $|g'(s)| = |1/2 - 1/s^2| \le 1/2$, if $|s| \ge 1$ and hence

(B8) $$|g(x) - g(y)| = \Big|\int_y^x g'(s)\,ds\Big| \le \int_y^x |g'(s)|ds \le \frac{|x - y|}{2}, \quad \text{if} \quad x \ge y \ge 1.$$

Moreover, if $x \ge 0$ then $g(x) \ge 1$ is equivalent to $x^2 + 2 - 2x = (x - 1)^2 + 1 \ge 0$ which is always true.