Math 150B. Calculus on Manifolds Practice Final, Spring 09

There will be several problems were you are asked to calculate the volume form on a surface, i.e. the surface area element. There are at least 3 ways to do this within the course that you should know how to do:

1) For a parameterized surface, as in the proof the theorem in the beginning of Section 8.3

2) For a level set using Corollary 8.15 and that the normal can be calculated as in Example 8.9.

3) For a graph as in Problems 8.6-7.

The problems on the practice midterm and midterm.

From the lecture notes: problems 6.13, 6.14, 6.15, 8.6, 8.7, 9.2, 9.3, 9.4.

In addition the following problems:

1. Let $\alpha = \sum_{i=1}^{n} f_i dx_i$ be a 1-form on $\mathbf{R}^n$.

(a) Find formulas for $*\alpha$, $d{*}\alpha$, $*d{*}\alpha$, and $d{*}d{*}\alpha$.
(b) Find formulas for $d\alpha$, $*d\alpha$, $d{*}d\alpha$, and $*d{*}d\alpha$.
(c) Finally compute $d{*}d{*}\alpha + (-1)^n {*}d{*}d\alpha$. Try to write the answer in terms of the Laplace operator Δ, which is given by $\Delta f = \sum_{i=1}^{n} \partial^2 f / \partial x_i^2$.

2. Let $\psi\colon [a,b] \to \mathbf{R}^n$ be an embedding. Then $M = \psi([a,b])$ is a smooth 1-manifold. Let us call the direction of the tangent vector $\psi'(t)$ positive; this defines an orientation of M. Let μ be the element of arclength ("volume element") of M.

(a) Show that $\psi^*\mu = \|\psi'(t)\|\, dt = \sqrt{\psi_1'(t)^2 + \psi_2'(t)^2 + \cdots + \psi_n'(t)^2}\, dt$, where t denotes the coordinate on $\mathbf{R}$. Conclude that the arclength ("volume") of M is $\int_a^b \|\psi'(t)\|\, dt$.
(b) Compute the arclength of the astroid $x = \cos^3 t$, $y = \sin^3 t$, where $t \in [0, \pi/2]$.
(c) Consider a plane curve given in polar coordinates by an equation $r = f(\theta)$. Show that its element of arclength is $\sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$. (Apply the result of part (a) to $\psi(\theta) = \big(f(\theta)\cos\theta, f(\theta)\sin\theta\big)$.)
(d) Compute the arclength of the cardioid given by $r = 1 + \cos\theta$.

3. Let M be the set of points in $\mathbf{R}^2$ given by the equation $(x^2+y^2)^2 + y^2 - x^2 = 0$.

(a) Show that $M - \{(0,0)\}$ is a 1-manifold.
(b) Determine the points where M has horizontal or vertical tangent lines.
(c) Sketch M. (Start by finding the intersection points of M with an arbitrary line through the origin, $y = ax$.)
(d) Is M a manifold at $(0,0)$? Explain.

4. Let $\alpha = (2xyz - x^2y)\, dy\, dz + (xz^2 - y^2z)\, dz\, dx + (2xyz - xy^2)\, dx\, dy$.

(a) Check that α is closed.
(b) Find a 1-form β such that $d\beta = \alpha$. You may use any method, for example the following. Let $\phi(x,y,z,t) = (tx, ty, tz)$ be the radial contraction, where $0 \le t \le 1$; then $\beta = \kappa(\phi^*\alpha)$ will do the job. Here κ is the operator that turns k-forms on $\mathbf{R}^3 \times [0,1]$ into $k-1$-forms on $\mathbf{R}^3$ as follows: if a k-form μ on $\mathbf{R}^3 \times [0,1]$ does not involve dt, then $\kappa(\mu) = 0$, and if $\mu = g(x,t)\, dt\, dx_J$, then $\kappa(\mu) = \big(\int_0^1 g(x,t)\, dt\big) dx_J$.