Math 150B. Calculus on Manifolds Practice Midterm, Spring 09, Lindblad.

1. Let $\alpha = x_1\, dx_2 + x_3\, dx_4$, $\beta = x_1 x_2 dx_3 \wedge dx_4 + x_3 x_4 dx_1 \wedge dx_2$ and $\gamma = x_2 dx_1 \wedge dx_3 \wedge dx_4$ be forms on $\mathbf{R}^4$. Calculate
   (a) $\alpha \wedge \beta$ and $\alpha \wedge \gamma$.
   (b) $d\beta$ and $d\gamma$.
   (c) $*\alpha$ and $*\gamma$.

2. Let α be the 1-form $\alpha = x_2 dx_1 + x_1 dx_2$ on $\mathbf{R}^2$.
   (a) Check that α is closed.
   (b) Find a function g such that $dg = \alpha$.

3. Define a map $\phi : \mathbf{R}^2 \to \mathbf{R}^4$ by

$$(y_1, y_2, y_3, y_4) = \phi(x_1, x_2) = \big(x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3\big).$$

   (a) Find $\phi^*(y_1 + 3y_2 + 3y_3 + y_4)$.
   (b) Find $\phi^* dy_1$ and $\phi^* dy_2$.
   (c) Find $\phi^*(dy_2 \wedge dy_3)$.

4. Define a curve $c : [0, \pi/2] \to \mathbf{R}^2$ by $c(t) = (a\cos t, b\sin t)$, where a and b are positive constants. Let $\alpha = xy\, dx + x^2 y dy$. Find $\int_c \alpha$ (for arbitrary a and b).

5. Define a 2-cube $c : [0,1]^2 \to \mathbf{R}^3$ by $c(t_1, t_2) = \big(t_1^2, t_1 t_2, t_2^2\big)$, and let $\alpha = x_1\, dx_2 + x_1 dx_3 + x_2\, dx_3$. Calculate both $\int_c d\alpha$ and $\int_{\partial c} \alpha$ and check that they are equal.