Math 168A Selected Homework 1 Solutions

C.T. Wildman

1.1.5 (a)

$$\begin{aligned}\mathbf{x} = (x_1, \dots, x_n) &= x_1(1, 0, \dots, 0) + x_2(0, 1, 0, \dots, 0) + \cdots + x_n(0, \dots, 0, 1) \\ &= \sum_{j=1}^{n} x_j \mathbf{e}_j\end{aligned}$$

(b) Observe that by part (a), we have for any $\mathbf{x} \in \mathbb{R}^n$, $f(\mathbf{x}) = f\left(\sum^n x_j \mathbf{e}_j\right) = \sum^n x_j f(\mathbf{e}_j)$ by linearity. This suggests that our $\mathbf{y}_f$ should be defined as $y_f = (f(\mathbf{e}_1), f(\mathbf{e}_2), \dots, f(\mathbf{e}_n))$. Note that this does not depend on $\mathbf{x}$. Then by definition of the dot product, we have $f(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y}_f \rangle$ as desired.

(c) First we will prove that if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{y} = 0$. To see this, we simply suppose $\mathbf{y} = \sum^n y_j \mathbf{e}_j$ and show that $y_j = 0$ for each j. This is immediate, since by assumption we have $0 = \langle \mathbf{e}_j, \mathbf{y} \rangle = y_j$ for each j.

To see that this is equivalent to uniqueness of $\mathbf{y}_f$ above, suppose there was another vector z_f such that $f(\mathbf{x}) = \langle \mathbf{x}, \mathbf{z}_f \rangle$ for all $\mathbf{x} \in \mathbb{R}^n$. Then it would follow that $0 = f(\mathbf{x}) - f(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y}_f \rangle - \langle \mathbf{x}, \mathbf{z}_f \rangle = \langle \mathbf{x}, \mathbf{y}_f - \mathbf{z}_f \rangle$ so that $\mathbf{y}_f - \mathbf{z}_f = 0$, i.e. $\mathbf{y}_f = \mathbf{z}_f$.

1.1.11 This problem is simply stating that, for fixed $0 \neq \mathbf{a} \in \mathbb{R}^n$, the solution space of the equation $\langle \mathbf{a}, \mathbf{x} \rangle = 0$ is an $n-1$ dimensional vector space.

We start by assuming that $\mathbf{a} = (a_1, \dots, a_n)$ and considering the above equation. We are looking for $x_1, \dots, x_n$ such that

$$a_1 x_1 + \cdots + a_n x_n = 0$$

This is one equation in n unknowns, so we are free to specify $n-1$ of the x_j. Without loss of generality we will assume that $a_n \neq 0$. Then it follows that

$$x_n = (-a_1/a_n)x_1 - \cdots - (a_{n-1}/a_n)x_{n-1}$$

A basis for the solution space (i.e. the $n-1$ linearly independent vectors the question asks for) is then given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ -\frac{a_1}{a_n} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ -\frac{a_2}{a_n} \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ -\frac{a_{n-1}}{a_n} \end{bmatrix} \right\}$$

1.1.12 Simply use induction and apply the result of 1.1.11.

1.2.1 We must show that $l_{t,\omega}$, which we have defined as $\{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, \omega \rangle = t\}$, is actually the same set as $\{t\omega + s\hat{\omega} \mid s \in \mathbb{R}\}$. To this end, take any point $\mathbf{x} = (x, y) \in l_{t,\omega}$. We would like to write $\mathbf{x} = t\omega + s\hat{\omega} = (t\omega_1 - s\omega_2, t\omega_2 + s\omega_1)$ for some s. Recall that $t = \langle \mathbf{x}, \omega \rangle$. We claim that $s = \langle \mathbf{x}, \hat{\omega} \rangle = y\omega_1 - x\omega_2$ satisfies $\mathbf{x} = t\omega + s\hat{\omega}$. A simple calculation verifies this.

Conversely, we must show that any vector of the form $t\omega + s\hat{\omega}$ lies on $l_{t,\omega}$. However, this is easy since $\langle t\omega + s\hat{\omega}, \omega \rangle = \langle t\omega, \omega \rangle + \langle s\hat{\omega}, \omega \rangle = t\langle \omega, \omega \rangle = t$ (remember ω and $\hat{\omega}$ are orthogonal).

1.2.7 From the previous exercise, we know that given (x, y), we have $t = x\omega_1 + y\omega_2$ and $s = \omega_1 y - \omega_2 x$. The determinant of the Jacobian matrix of this transformation is $\omega_1^2 + \omega_2^2 = 1$, so we have $dx\,dy = dt\,ds$.