Math 168A Selected Homework 2 Solutions

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2.1.1 Suppose $n \neq m$. Then either A or A^T has more columns than rows, and then either the equation $A\mathbf{x} = \mathbf{0}$ or the equation $A^T\mathbf{x} = \mathbf{0}$ is underdetermined and hence has a nontrivial solution. Thus either ker $A \neq \mathbf{0}$ or ker $A^T \neq \mathbf{0}$.

The converse is false. Consider, for example, the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It is 2×2 but clearly has a nontrivial kernel.

2.1.3 We must show that $A^{-1}(\mathbf{x} + \alpha\mathbf{y}) = A^{-1}\mathbf{x} + \alpha A^{-1}\mathbf{y}$ for any scalar α and vectors $\mathbf{x}, \mathbf{y}$. To this end, let $\mathbf{z} = A^{-1}\mathbf{x} + \alpha A^{-1}\mathbf{y}$. Then $A\mathbf{z} = A(A^{-1}\mathbf{x} + \alpha A^{-1}\mathbf{y}) = \mathbf{x} + \alpha\mathbf{y}$ since A is linear and $AA^{-1} = I = A^{-1}A$. It then follows that $\mathbf{z} = A^{-1}(\mathbf{x} + \alpha\mathbf{y})$, which is what was to be shown.

2.1.10 The limit matrix is $B = I + \sum_{j=1}^{\infty}(-1)^j A^j$, where we define $A_k = I + \sum_{j=1}^{k}(-1)^j A^j$. We compute

$$\|A_k - B\| = \left\| \sum_{j=k+1}^{\infty} (-1)^j A^j \right\| \leq \sum_{j=k+1}^{\infty} \|(-1)^j A^j\| \leq \sum_{j=k+1}^{\infty} \|A\|^j \to 0$$

as $k \to \infty$ (the tail sum of a convergent series converges to 0). Thus $B = \lim_k A_k$ in the sense of the norm. Now we can show that $(I + A)B = B(I + A) = I$. We compute

$$(I + A)B = (I + A)\lim_k A_k = \lim_k [(I + A)A_k] = \lim_k \left[I + A + \sum_{j=1}^{k}(-1)^j (A^j + A^{j+1}) \right]$$

Now we notice that the sum telescopes, and we are left only with

$$\lim_k [I + A - A + (-1)^k A^{k+1}] = \lim_k [I + (-1)^k A^{k+1}] = I$$

since $\lim_k A^{k+1} = 0$ (remember $\|A^{k+1}\| \leq \|A\|^{k+1} \to 0$, since $\|A\| < 1$).

A similar calculation shows that $B(I + A) = I$ so we are done.

2.2.2 We must show that $f \in \mathcal{C}^0[0,1]$, i.e. that f is continuous. In other words, for arbitrary $x \in [0,1]$, given $\epsilon > 0$, we must find $\delta > 0$ such that if $|x-y| < \delta$, then $|f(x)-f(y)| < \epsilon$.

So fix $x \in [0,1]$ and let $\epsilon > 0$ be given. First of all, since $\|f_k - f\|_\infty \to 0$, we can find some N such that $\|f_N - f\|_\infty < \epsilon$. Since every f_k is continuous at x, we know f_N is as well, so we can find some $\delta > 0$ such that if $|x-y| < \delta$, $|f_N(x) - f_N(y)| < \epsilon$. Then we estimate, for $|x-y| < \delta$:

$$\begin{aligned}
|f(x) - f(y)| &= |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\
&\leq \|f_N - f\|_\infty + \epsilon + \|f_N - f\|_\infty \\
&< 3\epsilon
\end{aligned}$$

Since ϵ was arbitrary, we are done.

2.2.3 No, the conclusion is false if $\|\cdot\|_\infty$ is replaced by $\|\cdot\|_1$. Consider $f_k(x) = x^k$ and

$$f(x) = \begin{cases} 0 & 0 \leq x < 1, \\ 1 & x = 1. \end{cases}$$

Then f is discontinuous at $x = 1$ yet $\|f_k - f\|_1 \to 0$ as $k \to \infty$.

2.2.5 This makes sense a priori since we are integrating an even (resp. odd) periodic function over an interval with length equal to one period, so we expect to get 0. We compute, for $f_j(t) = \cos\left(\frac{\pi j t}{\delta}\right)$,

$$\begin{aligned}
M_\delta(f_j)(x) &= \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} \cos\left(\frac{\pi j t}{\delta}\right) \, dt \\
&= \frac{1}{2\pi j} \sin\left(\frac{\pi j t}{\delta}\right) \Bigg|_{x-\delta}^{x+\delta} = 0
\end{aligned}$$

since $\sin\left(\frac{\pi j x}{\delta} + \pi j\right) = \sin\left(\frac{\pi j x}{\delta} - \pi j\right)$