Math 168A Selected Homework 3 Solutions

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3.1.1 Following the calculations on p. 59, we see that by conservation of energy we must have

$$I_0 = \int_{\|\mathbf{x}\|=r} I(r)\, dS = I(r) \int_{\|\mathbf{x}\|=r} dS = 4\pi r^2 I(r)$$

Therefore

$$I(r) = \frac{I_0}{4\pi r^2}$$

3.4.2 Fix $\mathbf{a} \in \mathbb{R}^2$. We will use the formula for the Radon transform of $g = \chi_{B_1}$ to compute the transform of $f = \chi_{B_1(a)}$. We have

$$\mathfrak{R}g(t,\omega) = \begin{cases} 2\sqrt{1-t^2} & \text{if } |t| \le 1 \\ 0 & \text{if } |t| > 1 \end{cases}$$

Now we observe that $f(\mathbf{x}) = g(\mathbf{x} - \mathbf{a})$ and compute:

$$\mathfrak{R}f(t,\omega) = \int_{-\infty}^{\infty} f(s\hat{\omega} + t\omega)\, ds = \int_{-\infty}^{\infty} g(s\hat{\omega} + t\omega - \mathbf{a})\, ds$$

At this point, we write $\mathbf{a} = \langle \mathbf{a}, \omega\rangle\omega + \langle \mathbf{a}, \hat{\omega}\rangle\hat{\omega}$ and get

$$\int_{-\infty}^{\infty} g\left[(s - \langle \mathbf{a}, \hat{\omega}\rangle)\hat{\omega} + (t - \langle \mathbf{a}, \omega\rangle)\omega\right]\, ds$$

Changing variables to $u = s - \langle \mathbf{a}, \hat{\omega}\rangle$ gives us

$$\int_{-\infty}^{\infty} g(u\hat{\omega} + (t - \langle \mathbf{a}, \omega\rangle)\omega)\, du = \mathfrak{R}g(t - \langle \mathbf{a}, \omega\rangle, \omega)$$

Thus, we have shown that

$$\mathfrak{R}f(t,\omega) = \begin{cases} 2\sqrt{1-(\langle \mathbf{a}, \omega\rangle)^2} & \text{if } |t - \langle \mathbf{a}, \omega\rangle| \le 1 \\ 0 & \text{if } |t - \langle \mathbf{a}, \omega\rangle| > 1 \end{cases}$$

3.4.4 Notice that $f(x, y)$ depends only on $r = \sqrt{x^2 + y^2}$ so we can use the formula for the Radon transform of radially symmetric functions. We have $f(x, y) = F(r)$, where $F(r) = \begin{cases} 1 & \text{if } r = 1 \\ 0 & \text{else} \end{cases}$ Then

$$\mathfrak{R}f(t, \omega) = 2 \int_{|t|}^{\infty} \frac{rF(r)}{\sqrt{r^2 - t^2}}\, dr = 0$$

since F is 0 everywhere except at $r = 1$.

3.4.5 Suppose $f(\mathbf{x}) \geq g(\mathbf{x})$ for all $\mathbf{x}$. Then $(f - g)(\mathbf{x}) \geq 0$ for all $\mathbf{x}$. Then by Proposition 3.4.1, we have $\mathfrak{R}(f - g)(t, \omega) \geq 0$. Then by linearity of the transform, we get $\mathfrak{R}f(t, \omega) - \mathfrak{R}g(t, \omega) \geq 0$, which is the same as $\mathfrak{R}f(t, \omega) \geq \mathfrak{R}g(t, \omega)$.

3.4.9 For simplicity, we will show that the segment $[0, 1]$ has measure 0 as a subset of the plane. Let $\epsilon > 0$ be given, and choose N large enough so that the following inequality holds: $\frac{1}{N} + \frac{1}{N^2} < \epsilon$. For $j = 0, 1, \ldots, N$, let $x_j = \frac{j}{N}$ and let $r_j = \frac{1}{N}$ for all j. Then the segment $[0, 1]$ is contained in the union $\bigcup_{j=0}^{N} B_{r_j}(x_j)$ and we have

$$\sum_{j=0}^{N} r_j^2 = \sum_{j=0}^{N} \frac{1}{N^2} = \frac{N+1}{N^2} < \epsilon$$

3.4.12 By completeness of the space $L^1(\mathbb{R}^2)$, it is enough to show that the sequence $\{\mathfrak{R}f_n\}$ is Cauchy in L^1.. We have

$$\|\mathfrak{R}f_n - \mathfrak{R}f_m\|_1 = \|\mathfrak{R}(f_n - f_m)\|_1 \leq \|\mathfrak{R}(f_n - f_m)\|_{1,\infty} \leq \|f_n - f_m\|_1 \to 0$$

where the second inequality follows from Proposition 3.4.4. The last quantity goes to 0 since the sequence $\{f_n\}$ converges in L^1.

3.4.13 We have $f(x, y) = F(r)$, where $F(x) = \frac{1}{x}\chi_{[0,1]}(x^2)$. Thus we can use the radially symmetric formula for the transform:

$$\mathfrak{R}f(t, \omega) = 2 \int_{|t|}^{\infty} \frac{rF(r)}{\sqrt{r^2 - t^2}}\, dr = 2 \int_{|t|}^{\infty} \frac{\chi_{[0,1]}(r^2)}{\sqrt{r^2 - t^2}}\, dr$$

We will get 0 if $|t| > 1$; otherwise we get

$$2 \int_{|t|}^{1} \frac{1}{\sqrt{r^2 - t^2}}\, dr = 2 \log \left| r + \sqrt{r^2 - t^2} \right|_{|t|}^{1} = 2 \log \frac{1 + \sqrt{1 - t^2}}{|t|}$$

Summarizing, we have

$$\mathfrak{R}f(t, \omega) = \begin{cases} 2 \log \frac{1 + \sqrt{1 - t^2}}{|t|} & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1 \end{cases}$$