# Math 168A Selected Homework 4 Solutions

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4.1.5 Suppose Re $a < 0$. Then we have

$$\left|\int_0^\infty e^{ax}\,dx\right| \le \int_0^\infty |e^{ax}|\,dx = \int_0^\infty \left|e^{(\text{Re }a + i\text{Im }a)x}\right|\,dx = \int_0^\infty e^{(\text{Re }a)x}\,dx = -\frac{1}{\text{Re }a} = \frac{1}{|\text{Re }a|}$$

Note that we only evaluated the definite integral of a real-valued function.

4.2.1 Suppose $f$ is continuous and in $L^1(\mathbb{R})$. It follows that $f$ is bounded on $(-\infty, \infty)$, say $|f| \le M$. Notice that by continuity, we have $f(x - 2\sqrt{\epsilon}t) \to f(x)$ as $\epsilon \to 0$. Thus $|f(x - 2\sqrt{\epsilon}t)e^{-t^2}| \le Me^{-t^2} \in L^1(\mathbb{R})$ so we can apply the Dominated Convergence Theorem to obtain

$$\lim_{\epsilon\to 0^+} \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty f(x - 2\sqrt{\epsilon}t)e^{-t^2}\,dt = \frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty f(x)e^{-t^2}\,dt = \frac{f(x)}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-t^2}\,dt = f(x)$$

4.2.2 Suppose $f \in L^1(\mathbb{R})$. That $\hat{f}$ is bounded is clear since

$$|\hat{f}(\xi)| = \left|\int f(x)e^{-ix\xi}\,dx\right| \le \int |f(x)|\,dx = \|f\|_1 < \infty$$

For continuity, fix any $\xi \in \mathbb{R}$ and compute (we will use the Dominated Convergence Theorem since $|f(x)e^{-ixt}| \le |f(x)| \in L^1$)

$$\lim_{t\to\xi} \hat{f}(t) = \lim_{t\to\xi}\int_{-\infty}^\infty f(x)e^{-ixt}\,dx = \int_{-\infty}^\infty f(x)e^{-ix\xi}\,dx = \hat{f}(\xi)$$

Therefore $\hat{f}$ is continuous at $\xi$, and since $\xi$ was arbitrary, we are done.

4.2.4 Let $f(x) = \hat{\chi}_{[a,b)}(x)$. That $\hat{f}(\xi) = \frac{e^{-ia\xi} - e^{-ib\xi}}{i\xi}$ is easily shown. To derive the bound, we first write the transformed function in terms of sines and cosines using the identities for $\sin\theta \pm \sin\phi$, etc. Recall that we are assuming $a < b$. We get

$$\begin{aligned}
\hat{f}(\xi) &= \frac{-2\sin(\frac{a+b}{2}\xi)\sin(\frac{a-b}{2}\xi) + 2i\sin(\frac{a-b}{2}\xi)\cos(\frac{a+b}{2}\xi)}{i\xi} \\
&= -2\sin\left(\tfrac{a-b}{2}\xi\right)\frac{\sin(\frac{a+b}{2}\xi) - i\cos(\frac{a+b}{2}\xi)}{i\xi} \\
&= -2\sin\left(\tfrac{a-b}{2}\xi\right)\frac{ie^{i\frac{a+b}{2}\xi}}{i\xi} = -(a-b)\frac{\sin\left(\frac{a-b}{2}\xi\right)}{\left(\frac{a-b}{2}\xi\right)}e^{i\frac{a+b}{2}\xi}
\end{aligned}$$

For simplicity let $c = \frac{a-b}{2}$. What we have now is that $|\hat{f}(\xi)| = 2|c|\frac{|\sin(c\xi)|}{|c\xi|}$. Note that $\hat{f}(\xi) \to 2|c|$ as $\xi \to 0$. that First consider the case $|\xi| \leq 1$. On this set, we have $\frac{1}{1+|\xi|} \geq \frac{1}{2}$ so that $|\hat{f}| \leq \frac{4|c|}{1+|\xi|}$ there. Let $M_1 = 4|c| = 2|a-b|$. On the set $|\xi| > 1$, we notice that $|\hat{f}(\xi)| \leq \frac{2}{|\xi|}$ and let $M_2 = 4$. Then on this set $|\hat{f}(\xi)| \leq \frac{M_2}{1+|\xi|}$ since $|\xi|^{-1} \leq 2(1+|\xi|)^{-1}$ if $|\xi| > 1$.

Now we simply set $M = \max\{M_1, M_2\}$ and we have proven that $|\hat{f}(\xi)| \leq \frac{M}{1+|\xi|}$ for all $\xi$.

4.2.6 Denote by $f_a(x)$ the function $f(ax)$. We use the formula $\mathcal{F}(f_a)(\xi) = a^{-1}\mathcal{F}(f)(a^{-1}\xi)$ (which is easily checked). This gives

$$\mathcal{F}(e^{-ax^2})(\xi) = \mathcal{F}(e^{(\sqrt{a}x)^2})(\xi) = \frac{\sqrt{\pi}}{\sqrt{a}}e^{-\frac{\xi^2}{4a^2}} = \sqrt{\frac{\pi}{a}}e^{-\frac{\xi^2}{4a^2}}$$

4.2.18 For $f \in L^2$, recall that we define $\hat{f}$ as the $L^2$ limit of $\hat{f}_R(\xi) = \int_{-R}^{R} f(x)e^{-ix\xi}\,dx$ (in other words, we have $\|\hat{f} - \hat{f}_R\|_2 \to 0$). Now define $f_R(x) = \int_{-R}^{R} \hat{f}(\xi)e^{ix\xi}\,d\xi$. By Parseval's Formula, it follows that

$$\|f - f_R\|_2^2 = \frac{1}{2\pi}\|\hat{f} - \hat{f}_R\|_2^2 \to 0$$

so we immediately see that $f$ is the $L^2$ limit of $f_R$ as $R \to \infty$.

4.2.21 This is easy to prove for any $f$ for which some form of the Fourier Inversion formula holds. If this is the case, then we simply write

$$\mathcal{F}\left(\overline{\hat{f}}\right)(\xi) = \overline{\int_{-\infty}^{\infty} \hat{f}(x)e^{ix\xi}\,dx} = 2\pi\overline{f(\xi)}$$

Note that the book forgot to multiply by $2\pi$.