Math 168A Selected Homework 5 Solutions

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5.1.12 (a) shift invariant

We have $\mathcal{A}_\tau f = f_\tau$, and since $(f_\tau)_\sigma = (f_\sigma)_\tau$, we see that $(\mathcal{A}_\tau f)_\sigma = \mathcal{A}_\tau(f_\sigma)$ so that $\mathcal{A}_\tau$ is shift-invariant.

(b) not shift invariant

We compute $(\mathcal{A}_\epsilon f_\tau)(x) = \epsilon^{-n} f_\tau(\epsilon^{-1}x) = \epsilon^{-n} f(\epsilon^{-1}x - \tau)$ but on the other hand, $(\mathcal{A}_\epsilon f)_\tau(x) = \epsilon^{-n} f(\epsilon^{-1}x - \epsilon^{-1}\tau)$

(c) not shift invariant

We get $(\mathcal{A}_\psi f)_\tau(x) = (\psi f)_\tau(x) = \psi(x-\tau)f(x-\tau)$ but $(\mathcal{A}_\psi f_\tau)(x) = \psi(x) f_\tau(x) = \psi(x)f(x-\tau)$

5.1.15 If $f(x) = \chi_{[-1,1]}(x)(1-|x|)^2$, then we have

$$\begin{aligned}\hat{f}(\xi) = \int_{-1}^{1}(1-|x|)^2 e^{-i\xi x}\,dx &= \int_{-1}^{1}(1-|x|)^2\cos(\xi x)\,dx \\ &= 2\int_0^1 (1-x)^2\cos(\xi x)\,dx \\ &= \frac{4(\xi - \sin(\xi))}{\xi^3}\end{aligned}$$

(to evaluate the last integral, consult a table of integrals for integrands of the form $x^k\cos(\xi x)$).

Now notice that $\xi - \sin(\xi) = 0$ only when $\xi = 0$. Since $\hat{f}(\xi)$ has a nonzero limit at $\xi = 0$, we know that $\hat{f}$ never vanishes. It follows immediately that if $g = f *_j f$, then $\hat{g}$ doesn't vanish either since $\hat{g} = (\hat{f})^j$.

5.2.2 We must compute $\int_{-1}^{1}(1-x^2)^k\,dx = 2\int_0^1(1-x^2)^k\,dx$. By making the substitution $x=\sin(\theta)$, we obtain

$$\begin{aligned}
2\int_0^{\pi/2}\cos^{2k+1}(\theta)\,d\theta &= \left.\frac{2\cos^{2k}(\theta)\sin(\theta)}{2k+1}\right|_0^{\pi/2} + 2\cdot\frac{2k}{2k+1}\int_0^{\pi/2}\cos^{2k-1}(\theta)\,d\theta \\
&= 2\cdot\frac{2k}{2k+1}\left[\left.\frac{\cos^{2k}(\theta)\sin(\theta)}{2k-1}\right|_0^{\pi/2} + \frac{2k-2}{2k-1}\int_0^{\pi/2}\cos^{2k-3}(\theta)\,d\theta\right] \\
&= \frac{2(2k)!!}{(2k+1)!!}
\end{aligned}$$

Thus the required constants are $c_k = \frac{(2k+1)!!}{2(2k)!!}$.

5.2.3 The best way to do this is to approximate f by a continuous function of compact support. Given $\epsilon > 0$, choose $g \in C_c(\mathbb{R}^n)$ such that $\|f-g\|_1 < \epsilon/3$. Now

$$\lim_{\tau\to 0}\int_{\mathbb{R}^n}|g(x-\tau)-g(x)|\,dx = 0$$

since for sufficiently small τ we can bound $|g(x-\tau)-g(x)|$ on a compact set and use dominated convergence.

To get the result for f, suppose τ is small enough that $\|g_\tau - g\|_1 < \epsilon/3$ and use the estimate

$$\|f_\tau - f\|_1 \le \|f_\tau - g_\tau\|_1 + \|g_\tau - g\|_1 + \|g - f\|_1 < \epsilon$$

5.2.6 We estimate as follows:

$$\begin{aligned}
\|\psi_\epsilon * f - f\|_2^2 &= \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\psi_\epsilon(y)f(x-y)\,dy - f(x)\right|^2\,dx \\
&= \int_{-\infty}^{\infty}\left|\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}f(x-y)\,dy - f(x)\right|^2\,dx \\
&= \int_{-\infty}^{\infty}\left|\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}f(x-y)-f(x)\,dy\right|^2\,dx
\end{aligned}$$

It now follows from Minkowski's integral inequality that

$$\|\psi_\epsilon * f - f\|_2 \le \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\left[\int_{-\infty}^{\infty}|f(x-y)-f(x)|^2\,dx\right]^{1/2} = \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\|f_y - f\|_2\,dy$$

The result we proved in the previous problem holds also in L^2, so if we choose ϵ appropriately, the last quantity in our estimate will be arbitrarily small.

5.3.1 Recall that a function that is both continuous and in $L^1(\mathbb{R})$ has to be bounded on $(-\infty, \infty)$. Now we compute

$$(\phi_\epsilon * f)(x) = \int_{-\infty}^{\infty} \phi_\epsilon(y) f(x-y)\, dy = \int_{-\infty}^{\infty} \epsilon^{-1}\phi(\epsilon^{-1}y) f(x-y)\, dy$$

Changing variables to $u = \epsilon^{-1}y$, we get

$$\int_{-\infty}^{\infty} \phi(u) f(x - \epsilon u)\, du$$

Now by continuity of f and dominated convergence, as $\epsilon \to 0$ we get that this last quantity tends to $f(x)\int_{\mathbb{R}} \phi = f(x)$.

6.1.1 Suppose f is piecewise continuous with bounded support. For fixed ω, we can always write $\mathbf{v} = \langle \omega, \mathbf{v}\rangle \hat{\omega} + \langle \omega, \mathbf{v}\rangle \omega$. Then we compute

$$\begin{aligned}
\mathfrak{R} f_{\mathbf{v}}(t, \omega) = \int_{-\infty}^{\infty} f_{\mathbf{v}}(s\hat{\omega} + t\omega)\, ds &= \int_{-\infty}^{\infty} f(s\hat{\omega} + t\omega - \mathbf{v})\, ds \\
&= \int_{-\infty}^{\infty} f((s - \langle \omega, \mathbf{v}\rangle)\hat{\omega} + (t - \langle \omega, \mathbf{v}\rangle)\omega)\, ds \\
&= \int_{-\infty}^{\infty} f(u\hat{\omega} + (t - \langle \omega, \mathbf{v}\rangle)\omega)\, du \\
&= \mathfrak{R} f(t - \langle \omega, \mathbf{v}\rangle, \omega)
\end{aligned}$$

(we made the change of variables $u = s - \langle \omega, \mathbf{v}\rangle$)

4.3.10 We have the relation $\int_{-\infty}^{\infty} f'(x)g(x)\, dx = \int_{-\infty}^{\infty} f(x)g'(x)\, dx$ for any $g \in C_c^1(\mathbb{R})$. The idea is to choose $g_n \in C_c^1(\mathbb{R})$, supported in $\{|x| > R\}$ such that $g_n \to \chi_{\{|x|>R\}} f'$ (in the L^2 sense). We may also assume (by passing to a subsequence) that the convergence is pointwise. Then by dominated convergence, we obtain

$$\int_{|x|>R} |f'(x)|^2\, dx = \lim_n \int_{-\infty}^{\infty} f'(x) g_n(x)\, dx = -\lim_n \int_{-\infty}^{\infty} f(x) g_n'(x)\, dx = 0$$

since the g_n' are also supported in $\{|x| > R\}$. It now follows that f' vanishes if $|x| > R$.