Math 168A Selected Homework 6 Solutions

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6.3.1 With the given assumptions on f and g, we can say that $f, g \in L^1 \cap L^2$. This allows us to simply write

$$\mathcal{H}(f * g) = \mathcal{F}^{-1}(\operatorname{sgn} \widehat{f * g}) = \mathcal{F}^{-1}(\operatorname{sgn} \hat{f}\hat{g}) = \mathcal{F}^{-1}(\operatorname{sgn} \hat{f}) * g = (\mathcal{H}f) * g$$

The other equality is similar.

6.3.2 An appropriate modification of the argument in the book p.203 (for the case of Hölder continuous functions of compact support) is all that is necessary here (instead of compact support we have that f is in L^2).

6.3.4 Recall that if we define $\tilde{f}(x) = f(-x)$, then we have that $\mathcal{F}\left[\tilde{\hat{f}}\right] = f$.

Now if $f(t) = e^{-\epsilon t}\chi_{[0,\infty)}(t)$ and $g(t) = e^{\epsilon t}\chi_{(-\infty,0]}(t)$, then we have

$$\hat{f}(x) = \frac{1}{\epsilon + ix}$$
$$\hat{g}(x) = \frac{1}{\epsilon - ix}$$

Now we simply observe that $a(x) := (x+i\epsilon)^{-1} = -i\hat{g}(x)$ and $b(x) := (x-i\epsilon)^{-1} = i\hat{f}(x)$. It follows that $-i\tilde{\hat{g}} = \tilde{a} = -b$ and $i\tilde{\hat{f}} = \tilde{b} = -a$. By taking Fourier transforms of both sides of these two equations, we obtain that

$$\hat{a}(\xi) = -if(\xi) = -ie^{-\epsilon\xi}\chi_{[0,\infty)}(\xi)$$
$$\hat{b}(\xi) = ig(\xi) = ie^{\epsilon\xi}\chi_{(-\infty,0]}(\xi)$$

6.3.5 1. Note that $\xi \operatorname{sgn}(\xi) = |\xi|$ and $\operatorname{sgn}(\xi)\operatorname{sgn}(\xi) = 1$ everywhere except 0. Thus, we have

$$A_1 f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\xi|^3 \hat{f}(\xi) e^{ix\xi}\, d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi^3 \hat{f}(\xi)\operatorname{sgn}(\xi) e^{ix\xi}\, d\xi$$
$$= i\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f'''}(\xi)\operatorname{sgn}(\xi) e^{ix\xi}\, d\xi$$

since $-i\xi^3\hat{f}(\xi) = (i\xi)^3\hat{f}(\xi) = \widehat{f'''}(\xi)$. It follows immediately that $A_1 f = i\mathcal{H}(\partial_x^3 f)$

6.3.7 3. Since $f \in L^2(\mathbb{R})$, we know that we can use Fourier inversion, etc. on f. Thus we can write

$$\mathcal{H}(\mathcal{H}f) = \mathcal{H}(\mathcal{F}^{-1}(\operatorname{sgn}\hat{f})) = \mathcal{F}^{-1}\left\{\operatorname{sgn}\mathcal{F}\left[\mathcal{F}^{-1}(\operatorname{sgn}\hat{f})\right]\right\} = \mathcal{F}^{-1}[\operatorname{sgn}^2\hat{f}] = \mathcal{F}^{-1}[\hat{f}] = f$$

6.8.1 Here I present a different method than the one from the practice midterm solutions.

We will compute $\mathcal{H}f(t)$ as $\lim_{\epsilon\to 0}(h_\epsilon * f)(t)$, where $h_\epsilon(t) = \frac{i}{\pi}\frac{t}{t^2+\epsilon^2}$, $f(t) = \chi_{[-1,1]}(t)$.

$$\begin{aligned}
(h_\epsilon * f)(t) = \frac{i}{\pi}\int_{-\infty}^{\infty} f(s)h_\epsilon(t-s)\,ds &= \frac{i}{\pi}\int_{-1}^{1}\frac{t-s}{(t-s)^2+\epsilon^2}\,ds \\
&= \frac{i}{\pi}\int_{t-1}^{t+1}\frac{u}{u^2+\epsilon^2}\,du \\
&= \frac{i}{\pi}\frac{1}{2}\log|u^2+\epsilon^2|\Big|_{t-1}^{t+1} \\
&= \frac{i}{\pi}\frac{1}{2}\log\left|\frac{(t+1)^2+\epsilon^2}{(t-1)^2+\epsilon^2}\right|
\end{aligned}$$

At this point we may evaluate the limit as $\epsilon \to 0$ to obtain

$$\mathcal{H}f(t) = \frac{i}{\pi}\frac{1}{2}\log\left|\frac{(t+1)^2}{(t-1)^2}\right| = \frac{i}{\pi}\log\left|\frac{t+1}{t-1}\right|$$

A.4.8 For $\phi \in C_0^\infty(\mathbb{R})$, recall that we can write $\frac{1}{x}(\phi(x)-\phi(0)) = \int_0^1 \phi'(xs)\,ds$

$$\begin{aligned}
|l_{1/x}(\phi)| &= \left|\int_{-1}^{1}\frac{\phi(x)-\phi(0)}{x}\,dx + \int_{|x|\geq 1}\frac{\phi(x)}{x}\,dx\right| \\
&\leq \int_{-1}^{1}\int_0^1 |\phi'(xs)|\,ds\,dx + \int_{|x|\geq 1}\frac{|\phi(x)|}{|x|}\,dx \\
&\leq 2\sup_{x\in\mathbb{R}}|\phi'(x)| + \int_{|x|\geq 1}\frac{(1+|x|)|\phi(x)|}{|x|(1+|x|)}\,dx \\
&\leq 2\sup_{x\in\mathbb{R}}|\phi'(x)| + 2\sup_{x\in\mathbb{R}}(1+|x|)|\phi(x)|\int_1^\infty\frac{1}{x(1+x)}\,dx \\
&= 2\sup_{x\in\mathbb{R}}|\phi'(x)| + (\log 4)\sup_{x\in\mathbb{R}}(1+|x|)|\phi(x)| \\
&\leq 2\sup_{x\in\mathbb{R}}(1+|x|)[|\phi(x)+|\phi'(x)|] = 2\|\phi\|_1
\end{aligned}$$

A.4.12 We have $\phi_n(x) = n(1 - |nx|)\chi_{[-1,1]}(nx) = n(1 - |nx|)\chi_{[-1/n,1/n]}(x)$, so we compute, for $f \in C_0^\infty$,

$$l_{\phi_n}(f) = n \int_{-1/n}^{1/n} (1 - |nx|) f(x)\, dx = \int_{-1}^{1} (1 - |u|) f(u/n)\, du \to f(0) = \delta(f)$$

by uniform convergence.

To see that $l_{\phi_n}^{[1]} \to \delta^{[1]}$, just do the same change of variables and use the definitions.