**168A. Math of Medical Imaging Practice Final, Fall 09, Lindblad.**

1. Compute the Radon transform $\mathcal{R}f(t,\omega)$ of $f(x,y) = xy\,\chi_B(x,y)$, where

$$B = B_1(0) = \{(x,y) \in \mathbf{R}^2;\ x^2+y^2<1\}.$$

Note: This function is not radially symmetric! Your answer will depend on $t$ and $\omega$!

2. (a) What condition on $f$ is required for $\widehat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)\, e^{ix\xi}\, dx$ to make sense?

(b) How can the Fourier transform be defined if we only know that $f \in L^2$?

3. Let $A_\theta$ denote rotation through an angle $\theta$. Set $\omega(\theta) = (\cos\theta, \sin\theta)$ and let $f_\theta(x,y) = f(A_\theta(x,y))$, where $A_\theta$ is the matrix rotating $(x,y)$ an angle $\theta$.

(a) Show that $\mathcal{R}f_\theta(t,\omega(\phi)) = \mathcal{R}f(t,\omega(\phi+\theta))$.

(b) Use (a) to show that $\partial_\theta \mathcal{R}f(t,\omega(\theta)) = \mathcal{R}[\Omega f](t,\omega(\theta))$ where $\Omega f = (y\partial_x - x\partial_y)f$.

4. (a) Show that $\widehat{f * g} = \widehat{f}\ \widehat{g}$.

(b) Show that $h(x) = \frac{L}{\pi}\operatorname{sinc}(Lx)$ is the inverse Fourier transform of $\widehat{h}(\xi) = \chi_{[-L,L]}(\xi)$

(c) Show that convolution $\mathcal{B}f = h * f$, with $h$, as in (b) defines a lowpass filter. What is the bandwidth?

5. (a) Suppose $f, g \in L^2(\mathbf{R})$. Show that

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi$$

(b) Use (a) to show that

$$\int_{-\infty}^{\infty} \operatorname{sinc}(Lx - n\pi)\operatorname{sinc}(Lx - m\pi)\, dx = \begin{cases} \frac{\pi}{L} & m = n \\ 0 & m \neq n \end{cases}$$

(c) Suppose that $f, g$ are $L$-bandlimited. Use the Shannon-Whitaker interpolation formula and (b) to show that

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = \frac{\pi}{L}\sum_{n=-\infty}^{\infty} f(\tfrac{n\pi}{L})\overline{g(\tfrac{n\pi}{L})}.$$

6. Let $f$ be the $2\pi$-periodic function $f(x) = (2\pi - x)^2$, when $0 \le x < 2\pi$.

(a) Compute the Fourier series $\sum c_k e^{ikx}$ of $f$.

(b) In which sense do the partial sums $S_N = \sum_{k=-N}^{N} c_k e^{ikx}$ converge to $f$?

(c) In which sense do the Fejer means $F_N = \sum_{k=-N}^{N} c_k \Big(1 - \dfrac{|k|}{N+1}\Big) e^{ikx}$ converge to $f$?

7. Let $f(x) = \pi^2 - x^2$, when $|x| \le \pi$, and $f(x) = 0$, when $|x| > \pi$.

(a) Find the Fourier transform. $\widehat{f}(\xi)$.

The finite sample Fourier transform is $\widehat{f_s}(\xi) = \tau \sum_{j=-M}^{M} f(j\tau)\, e^{ij\tau\xi}$, where $\tau = \frac{\pi}{M}$.

(b) Show that there is a constant $C$ such that $|\widehat{f_s}(\xi) - \widehat{f}(\xi)| \le CM^{-2}$, for $|\xi| \le M$.
(Hint: Use the dual Poisson summation formula.)