# Math 168A Practice Final Solutions

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1. A line $l_{t,\omega}$ will not intersect the unit circle if $|t| > 1$ so we have $\mathfrak{R}f(t,\omega) = 0$ if $|t| > 1$. If $|t| \leq 1$, we see that the line $l_{t,\omega}$ intersects the unit circle at the points $s_0\hat{\omega} + t\omega$ and $s_1\hat{w} + t\omega$, where $s_0 = -\sqrt{1-t^2}$ and $s_1 = \sqrt{1-t^2}$. Thus, we have for $|t| \leq 1$

$$\begin{aligned}
\mathfrak{R}f(t,\omega) &= \int_{-\infty}^{\infty} f(t\omega_1 - s\omega_2, t\omega_2 + s\omega_1)\,ds \\
&= \int_{s_0}^{s_1} (t\omega_1 - s\omega_2)(t\omega_2 + s\omega_1)\,ds \\
&= \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} (t^2 - s^2)\omega_1\omega_2 + st(\omega_1^2 - \omega_2^2)\,ds \\
&= (2t^2\sqrt{1-t^2} - \tfrac{2}{3}(1-t^2)^{\frac{3}{2}})\omega_1\omega_2 \\
&= \tfrac{2}{3}(4t^2 - 1)\sqrt{1-t^2}\,\omega_1\omega_2
\end{aligned}$$

To summarize:

$$\mathfrak{R}f(t,\omega) = \begin{cases} \frac{2}{3}\omega_1\omega_2(4t^2-1)\sqrt{1-t^2} & |t| \leq 1 \\ 0 & |t| > 1 \end{cases}$$

2. (a) The integral in question will converge absolutely if, for instance, $f \in L^1(\mathbb{R})$.

   (b) For $R > 0$, we can define $\hat{f}_R(\xi) = \int_{-R}^{R} f(x)e^{-ix\xi}\,dx$. This function has an $L^2$ limit we will denote by $\hat{f}$ (i.e. $\|\hat{f}_R - \hat{f}\|_2 \to 0$ as $R \to \infty$). We take this as our definition of $\hat{f}$.

3. (a) Observe that $A_\theta\omega(\phi) = \omega(\phi + \theta)$. Then we calculate

$$\begin{aligned}
\mathfrak{R}f_\theta(t,\omega(\phi)) = \int_{-\infty}^{\infty} f_\theta(s\hat{\omega}(\phi) + t\omega(\phi))\,ds &= \int_{-\infty}^{\infty} f(sA_\theta\hat{\omega}(\phi) + tA_\theta\omega(\phi))\,ds \\
&= \int_{-\infty}^{\infty} f(s\hat{\omega}(\phi+\theta) + t\omega(\phi+\theta))\,ds \\
&= \mathfrak{R}f(t,\omega(\phi+\theta))
\end{aligned}$$

(b) By (a) we can write

$$\begin{aligned}\partial_\theta \mathfrak{R} f(t,\omega(\theta)) &= \lim_{\phi\to 0} \frac{\mathfrak{R} f(t,\omega(\theta+\phi)) - \mathfrak{R} f(t,\omega(\theta))}{\phi} \\ &= \lim_{\phi\to 0} \frac{\mathfrak{R} f_\phi(t,\omega(\theta)) - \mathfrak{R} f(t,\omega(\theta))}{\phi} \\ &= \lim_{\phi\to 0} \mathfrak{R}\left[\frac{f_\phi - f}{\phi}\right](t,\omega(\theta))\end{aligned}$$

If we show that $\frac{f_\phi - f}{\phi}$ converges to $y\partial_x f - x\partial_y f$ as $\phi \to 0$, then we can use the Dominated Convergence theorem to pass the limit through the Radon transform to obtain the result.

We can write $A_\phi$ in matrix form as $A_\phi = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$ and let $(x(\phi), y(\phi)) = A_\phi(x,y) = (x\cos\phi + y\sin\phi, -x\sin\phi + y\cos\phi)$. Then

$$\lim_{\phi\to 0} \frac{f_\phi(x,y) - f(x,y)}{\phi} = \partial_\phi f(x(\phi), y(\phi))\bigg|_{\phi=0} = y\partial_x f(x,y) - x\partial_y f(x,y)$$

so we are done.

4. (a) The following calculation is valid if, for instance, $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

$$\begin{aligned}\widehat{f*g}(\xi) = \int_{-\infty}^{\infty} (f*g)(x) e^{-ix\xi}\, dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y) g(y) e^{-ix\xi}\, dy\, dx \\ &= \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} f(x-y) e^{-ix\xi}\, dx\, dy \\ &= \int_{-\infty}^{\infty} g(y) e^{-iy\xi} \hat{f}(\xi)\, dy \\ &= \hat{f}(\xi)\hat{g}(\xi)\end{aligned}$$

(b) We compute

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} \chi_{[-L,L]}(\xi) e^{ix\xi}\, d\xi = \frac{1}{2\pi}\int_{-L}^{L} e^{ix\xi}\, d\xi = \frac{1}{2\pi i x}(e^{iL\xi} - e^{-iL\xi}) &= \frac{\sin(Lx)}{\pi x} \\ &= \frac{L}{\pi}\frac{\sin(Lx)}{Lx} \\ &= \frac{L}{\pi}\operatorname{sinc}(Lx)\end{aligned}$$

(c) Using (a) and (b), with $h(x) = \frac{L}{\pi}\operatorname{sinc}(Lx)$, we compute that

$$\widehat{\mathcal{B}f}(\xi) = \widehat{h*f}(\xi) = \hat{h}(\xi)\hat{f}(\xi) = \hat{f}(\xi)\chi_{[-L,L]}(\xi)$$

Thus $\mathcal{B}$ is an ideal lowpass filter with bandwidth $2L$.

5. (a) From the Parseval identity, we know that for $h \in L^2(\mathbb{R})$, we have $\int_{-\infty}^{\infty} |h(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{h}(\xi)|^2\, d\xi$. Applying this to $h = f + \alpha g$ and using the notation $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx$, we obtain

$$\alpha\langle f, g\rangle + \overline{\alpha}\langle g, f\rangle = \alpha\langle \hat{f}, \hat{g}\rangle + \overline{\alpha}\langle \hat{g}, \hat{f}\rangle$$

First let $\alpha = 1$ and then let $\alpha = i$ to conclude that the real and imaginary parts of $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx$ and $\langle \hat{f}, \hat{g}\rangle = \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi$ are equal.

(b) First we observe that the Fourier transform of $\operatorname{sinc}(Lx - n\pi) = \operatorname{sinc}(L(x - \frac{n\pi}{L}))$ is $\frac{\pi}{L}e^{-\frac{in\pi\xi}{L}}\chi_{[-L,L]}(\xi)$. Then using (a) we calculate

$$\begin{aligned}\int_{-\infty}^{\infty} \operatorname{sinc}(Lx - n\pi)\operatorname{sinc}(Lx - m\pi)\, dx &= \frac{1}{2\pi}\frac{\pi^2}{L^2}\int_{-\infty}^{\infty} e^{-\frac{in\pi\xi}{L}} e^{\frac{im\pi\xi}{L}}\chi_{[-L,L]}(\xi)\, d\xi \\ &= \frac{\pi}{2L^2}\int_{-L}^{L} e^{-\frac{in\pi\xi}{L}} e^{\frac{im\pi\xi}{L}}\, d\xi \\ &= \frac{\pi}{2L}\int_{-1}^{1} e^{-i(m-n)\pi u}\, du \\ &= \begin{cases} \frac{\pi}{L} & m = n \\ 0 & m \neq n \end{cases}\end{aligned}$$

(c) The Shannon-Whitaker formula asserts that $f(x) = \sum_{n=-\infty}^{\infty} f(\frac{n\pi}{L})\operatorname{sinc}(Lx - n\pi)$. The analogous expansion holds for $g$, so by direct substitution, we obtain (for sufficiently regular $f, g$)

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(\tfrac{n\pi}{L})\operatorname{sinc}(Lx - n\pi) \sum_{m=-\infty}^{\infty} \overline{g(\tfrac{m\pi}{L})}\operatorname{sinc}(Lx - m\pi)\, dx \\ &= \sum_{m,n=-\infty}^{\infty} f(\tfrac{n\pi}{L})\overline{g(\tfrac{m\pi}{L})}\int_{-\infty}^{\infty} \operatorname{sinc}(Lx - n\pi)\operatorname{sinc}(Lx - m\pi)\, dx \\ &= \frac{\pi}{L}\sum_{n=-\infty}^{\infty} f(\tfrac{n\pi}{L})\overline{g(\tfrac{m\pi}{L})}\end{aligned}$$

6. (a) Some calculations give (for $k \neq 0$)

$$c_k = \frac{4\pi(i + k\pi)}{ik^2} = \frac{4\pi}{k^2} + \frac{4\pi^2}{ik}$$

We then compute $c_0 = \frac{4\pi^2}{3}$ so we obtain the Fourier series representation

$$\frac{4\pi^2}{3} + \sum_{k\neq 0} \frac{2(i + k\pi)}{ik^2} e^{ikx}$$

(b) We know that at least the partial sums converge to $f$ in $L^2([0,2\pi])$ since $f \in L^2([0,2\pi])$. However, they do not converge uniformly or even pointwise. For instance, at $x=0$, we have $f(x) = 4\pi^2$ but the value of the series is

$$\frac{4\pi^2}{3} + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{2}{k^2} + \frac{2\pi}{ik} = \frac{4\pi^2}{3} + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{2}{k^2}$$

As $N \to \infty$, we obtain $\frac{4\pi^2}{3} + \frac{4\pi^2}{6} = 2\pi^2$. This is expected since at a jump discontinuity, the Fourier series still converges to a value of half the height of the jump.

(c) By following the argument on page 264 of the text, we can show that the Fejer means $F_N$ converge to $f$ in $L^2([0,2\pi])$. Furthermore, we may apply Fejer's theorem (7.5.2) to conclude that the Fejer means converge to $f$ pointwise on $[0,2\pi)$.

7. (a) A calculation gives

$$\hat{f}(\xi) = \frac{4\sin(\pi\xi) - 4\pi\xi\cos(\pi\xi)}{\xi^3}$$

Note that $\hat{f} \in L^1(\mathbb{R})$ since there it has a finite limit as $\xi \to 0$ and decays fast enough at $\infty$. With part (b) in mind, we also make the preliminary estimate

$$|\hat{f}(\xi)| \leq 4\pi\frac{1+|\xi|}{|\xi|^3}$$

(b) We can apply the dual Poisson summation formula with $L = M$ to $f$ since $\hat{f}$ is integrable and $\sum_{n=-\infty}^{\infty} |f(\frac{n\pi}{M})| = \sum_{n=-M}^{M} |f(\frac{n\pi}{M})| < \infty$. The right hand side of the Poisson formula is then

$$\frac{\pi}{M}\sum_{n=-\infty}^{\infty} f(\tfrac{n\pi}{M})e^{-\frac{i\pi n\xi}{M}} = \frac{\pi}{M}\sum_{n=-M}^{M} f(\tfrac{n\pi}{M})e^{\frac{i\pi n\xi}{M}} = \hat{f}_s(\xi)$$

(where the equality follows since $f$ is an even function and vanishes outside $[-\pi,\pi]$). The left hand side of the Poisson formula is

$$\sum_{n=-\infty}^{\infty} \hat{f}(\xi + 2nM) = \hat{f}(\xi) + \sum_{n\neq 0} \hat{f}(\xi + 2nM)$$

Plugging into the Poisson formula then gives $\hat{f}_s(\xi) = \hat{f}(\xi) + \sum_{n\neq 0} \hat{f}(\xi + 2nM)$ so that we can estimate

$$|\hat{f}_s(\xi) - \hat{f}(\xi)| = \left|\sum_{n\neq 0} \hat{f}(\xi + 2nM)\right| \leq \sum_{n\neq 0} |\hat{f}(\xi + 2nM)|$$

By our estimate from part (a) and the assumption that $-M \leq \xi \leq M$, we see that (assuming WLOG that $M \geq 1$)

$$|\hat{f}(\xi + 2nM)| \leq 4\pi \frac{1 + |\xi + 2nM|}{|\xi + 2nM|^3} \leq 4\pi \frac{1 + |(2n+1)M|}{|(2n-1)M|^3} \leq 4\pi \frac{|2n+2|M}{|2n-1|^3 M^3}$$

Thus if we set $C = 4\pi \sum_{n \neq 0} \frac{|2n+2|}{|2n-1|^3} < \infty$, we have

$$|\hat{f}_s(\xi) - \hat{f}(\xi)| \leq \frac{C}{M^2}$$

as desired.