The Fourier transform. The Fourier transform $\mathcal{F}: f \to \widehat{f}$ is defined to be

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx \tag{4.1}$$

i.e. in n dimensions

$$\widehat{f}(\xi_1, \dots, \xi_n) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1, \cdots, x_n)\, e^{-i(x_1\xi_1 + \cdots + x_n\xi_n)}\, dx_1 \cdots dx_n.$$

The Fourier transform is invertible, and we will prove:

Theorem(Fourier's inversion formula).

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \widehat{f}(\xi)\, e^{ix\cdot\xi}\, dx. \tag{4.2}$$

We will directly prove the inversion formula in any number of dimensions n, but it might be easier to first think about the case $n = 1$. The Fourier transform makes sense for a very general class of functions and even distributions. However, it is natural to first define it for a more restrictive class and afterwards extend the definition by continuity to a larger class of functions. We first note that

Lemma 4.1. *If $f \in L^1$ then (3.1) defines a bounded (continuous) function $\widehat{f} \in L^\infty$:*

$$\|\widehat{f}\|_{L^\infty} = \sup_{\xi\in\mathbf{R}^n} |\widehat{f}(\xi)| \le \|f\|_{L^1} = \int_{\mathbf{R}^n} |f(x)|\, dx$$

Proof. By Minkowski's inequality (the triangle inequality for integrals)

$$\Big|\int_{\mathbf{R}^n} f(x) e^{-ix\cdot\xi}\, dx\Big| \le \int_{\mathbf{R}^n} \big|f(x) e^{-ix\cdot\xi}\big|\, dx = \int_{\mathbf{R}^n} |f(x)|\, dx. \quad \square$$

However, the inverse Fourier transform does not map L^∞ back to L^1. A more restrictive class that has this property is the Schwartz class $\mathcal{S}$ consisting of all infinitely differentiable functions that are rapidly decreasing: For all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ we assume that

$$\sup_x |x^\beta \partial^\alpha \phi(x)| < \infty, \tag{4.3}$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ and $\partial_j = \partial/\partial x_j$. In particular $e^{-|x|^2/2}$ is in this class. We have:

Lemma 4.2. *The Fourier transform maps the class of infinitely differentiable rapidly decreasing functions to itself; $\mathcal{F}: \mathcal{S} \to \mathcal{S}$. Moreover the following identities for the Fourier transform hold:*

$$\mathcal{F}: \partial_j f(x) \to i\xi_j \hat{f}(\xi), \qquad \mathcal{F}: x_j f(x) \to i\partial_j \hat{f}(\xi) \tag{4.4}$$

Proof. The first part of (4.4) follows from integrating by parts in (4.1)

$$\int \partial_j f(x) e^{-ix\cdot\xi}\, dx = \int \partial_j\big(f(x) e^{-ix\cdot\xi}\big)\, dx - \int f(x)\partial_j (e^{-ix\cdot\xi})\, dx = \int f(x) i\xi_j e^{-ix\cdot\xi}\, dx,$$

since if we first integrate in the x_j direction and use that f goes to 0 at ∞:

$$\int_{-\infty}^{+\infty} \partial_j\big(f(x) e^{-ix\cdot\xi}\big)\, dx_j = f(x) e^{-ix\cdot\xi}\Big|_{x_j=-\infty}^{x_j=+\infty} = 0.$$

The second part of (4.4) follows from differentiating below the integral sign in (4.1). That $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ now follows using (4.1) and integrating by parts

$$\xi^\alpha \partial_\xi^\beta \hat{\phi}(\xi) = \int \xi^\alpha (-i)^{|\beta|} x^\beta e^{-ix\cdot\xi} \phi(x)\, dx = (-1)^{|\alpha|}(-i)^{|\alpha|} \int \partial_x^\alpha\big(x^\beta \phi(x)\big)\, e^{-ix\cdot\xi}\, dx,$$

where $|\,\alpha| = \alpha_1 + \cdots + \alpha_n$. Here

$$\Big|\int \partial_x^\alpha\big(x^\beta\phi(x)\big)\, e^{-ix\cdot\xi}\, dx\Big| \le \int |\,\partial_x^\alpha\big(x^\beta\phi(x)\big)|dx \le \int \frac{C dx}{(1+|\,x|)^{n+1}},$$

since by (4.3) $\sup_x \big|\partial_x^\alpha\big(x^\beta\phi(x)\big)\big|(1+|x|)^{1+n} \le C$. This integral is easy bounded: □

Lemma 4.3. *Let c_n be the area of the unit sphere $\mathbf{S}^{n-1}$ in $\mathbf{R}^n$. Then*

$$\int \frac{dx}{(1+|\,x|)^{n+1}} \le 2c_n \tag{4.5}$$

Proof. To estimate the integral we use spherical coordinates i $\mathbf{R}^n$: Let $x = r\omega$, where $r = |\,x| > 0$ and $\omega = x/|\,x| \in \mathbf{S}^{n-1}$ belongs to the unit sphere. In spherical coordinates the volume form $dx = r^{n-1}dr\, dS(\omega)$. (If $n = 1$ there is nothing to prove since the 0 dimensional sphere consists of two points, if $n = 2$ this is the usual formula in polar coordinates $dx = rdrd\theta$ and if $n = 3$ it is the usual spherical coordinates $r^2 \sin\phi\, dr\, d\phi\, d\theta$.) In spherical coordinates we have:

$$\int \frac{dx}{(1+|\,x|)^{n+1}} = \int_0^\infty \int_{\mathbf{S}^{n-1}} \frac{r^{n-1} dS\, dr}{(1+r)^{n+1}} = \int_0^\infty \frac{c_n r^{n-1}\, dr}{(1+r)^{n+1}} \le c_n\Big(\int_0^1 dr + \int_1^\infty \frac{dr}{r^2}\Big) = 2c_n \quad \square$$

Note also that by changing variables we get two more simple properties:

Lemma 4.4.

$$\mathcal{F} : f(ax) \to a^{-n}\hat{f}(\xi/a), \qquad \mathcal{F} : f(x+h) \to \hat{f}(\xi)e^{ih\cdot\xi} \tag{4.6}$$

and

$$\mathcal{F} : f_1(x_1)\cdots f_n(x_n) \to \widehat{f_1}(\xi_1)\cdots \widehat{f_n}(\xi_n) \tag{4.7}$$

The proof of (4.2) uses:

Lemma 4.5.

$$\mathcal{F} : e^{-|ax|^2/2} \to (2\pi)^{n/2} a^{-n} e^{-|\xi/a|^2/2} \tag{4.8}$$

Proof. Using the previous lemma we can reduce to the case $a = 1$ and $n = 1$, since $e^{-|x|^2/2} = e^{-x_1^2/2} \cdots e^{-x_n^2/2}$. Let $\phi(x_1) = e^{-x_1^2/2}$. Since $(\partial_1 + x_1)\phi(x_1) = 0$ it follows that $(i\xi_1 + i\partial_1)\hat{\phi}(\xi_1) = 0$. This differential equation has the only solution $\hat{\phi}(\xi_1) = c_1 e^{-\xi_1^2/2}$, and it only remains to determine c_1. However, by (4.1) $c_1 = \hat{\phi}(0) = \int e^{-x_1^2/2}\, dx_1$. There is no anti derivative for e^{-x^2} in terms of elementary functions but we can calculate it with a trick. We have $c_1^2 = \int e^{-x_1^2/2}\, dx_1 \int e^{-x_2^2/2}\, dx_2 = \int\int e^{-(x_1^2+x_2^2)/2} dx_1 dx_2 = \int_{\mathbf{R}^2} e^{-|x|^2/2}\, dx$. This integral can easily be calculated by introducing polar coordinates $\int_{\mathbf{R}^2} e^{-|x|^2/2}\, dx = 2\pi \int_0^\infty e^{-r^2/2}\, r\, dr = 2\pi$. $\square$

Lemma 4.6. *If $f, g \in \mathcal{S}$ then*

$$\int f(\varepsilon x)\, g(x)\, dx \to f(0) \int g(x)\, dx \quad \varepsilon \to 0, \tag{4.9}$$

Proof. Since $|f(\varepsilon x)\, g(x)| \le \sup_y |f(y)|\, |g(x)|$ (4.9) follows from dominated converge. It is also easy to see directly; since $|f(\varepsilon x) - f(0)| \le \varepsilon |\, x|\, \sup_y |f'(y)| \le C\varepsilon |\, x|$ and $|\, x|\, |g(x)| \le C(1 + |x|)^{-1-n}$, the difference of the two sides of (4.8) is bounded by using Lemma 4.3:

$$\Big|\int f(\varepsilon x)\, g(x)\, dx - \int f(0)\, g(x)\, dx\Big| \le \int |f(\varepsilon x) - f(0)|\, |g(x)|\, dx \le \int \frac{C^2 \varepsilon dx}{(1 + |x|)^{1+n}} \le 2c_n C^2 \quad \square$$

We also have

Lemma 4.7.

$$\int \phi \widehat{\psi}\, d\xi = \int \widehat{\phi}\psi d\xi, \quad \phi, \psi \in \mathcal{S} \tag{4.10}$$

Proof. In fact, both sides of (4.10 are equal to the double integral

$$\int \hat{\phi}\psi\, d\xi = \int\int \phi(x) e^{-ix\cdot\xi}\, dx \psi(\xi)\, d\xi = \iint \phi(x)\psi(\xi)\, e^{-ix\cdot\xi}\, dx d\xi. \quad \square$$

It follows from using (4.6) that it suffices to prove (4.2) for $x = 0$ since its translation invariant, i.e. if (4.2) is true for all functions $f \in \mathcal{S}$ at $x = 0$ then its true for the translates $f_h(x) = f(x + h)$ at $x = 0$. Since by (4.6) $\widehat{f_h}(\xi) = \widehat{f}(\xi) e^{ih\cdot\xi}$;

$$f(h) = f_h(0) = \frac{1}{(2\pi)^n} \int \widehat{f_h}(\xi)\, d\xi = \frac{1}{(2\pi)^n} \int \widehat{f}(\xi)\, e^{ih\cdot\xi}\, d\xi.$$

Using (4.10) and (4.6) gives

$$\int \hat{\phi}(x)\, f(\varepsilon x)\, dx = \int \phi(x)\, f(\xi/\varepsilon) \varepsilon^{-n} d\xi = \int \phi(\varepsilon\eta)\, \hat{f}(\eta)\, d\eta$$

By Lemma 4.6 we get as $\varepsilon \to 0$

$$\int \hat{\phi}(x)\,dx\,f(0) = \phi(0)\int \hat{f}(\xi)\,d\xi$$

Picking $\phi(x) = e^{-|x|^2/2}$ we get from Lemma 4.5 and its proof that $\int \hat{\phi}(x)\,dx = (2\pi)^n$ and Fourier's inversion formula (4.2) follows for $x = 0$:

$$f(0) = \frac{1}{(2\pi)^n}\int \widehat{f}(\xi)\,d\xi$$

This concludes the proof of (4.2).

Note that the Fourier transform apart from a sign change or complex conjugation is its own inverse:

Lemma 4.8. *If $h = \overline{\widehat{g}}$, then $\widehat{h} = \overline{g}$.*

Using Fourier's inversion formula, (4.10) and the previous lemma we get

Corollary(Parseval's formula).

$$\int f(x)\overline{g(x)}\,dx = \frac{1}{(2\pi)^n}\int \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi \tag{4.11}$$

In particular;

$$\int |f(x)|^2\,dx = \frac{1}{(2\pi)^n}\int |\widehat{f}(\xi)|^2\,d\xi. \tag{4.12}$$

Using the previous lemma we can extend the Fourier transform by continuity to a map $\mathcal{F}: L^2 \to L^2$. This follows from:

Lemma 4.9. *For any $f \in L^2$ there is a sequence of functions $f_n \in \mathcal{S}$ such that $f_n \to f$ in L^2*

We will not prove this lemma here. Partly it will follow from the results in Chapter 5.

It follows from Lemma 4.9 that $\widehat{f_n} \in \mathcal{S}$ converges in L^2 to some function $\widehat{f} \in L^2$. In fact the sequence $\{f_n\}$ is Cauchy, i.e. $\|f_j - f_k\|_{L^2} \to 0$, as $j, k \geq N \to \infty$ so by the preceding lemma

$$\|\widehat{f_j} - \widehat{f_k}\|_{L^2} = (2\pi)^{n/2}\|f_j - f_k\|_{L^2} \to 0, \quad \text{as} \quad j, k \geq N \to \infty.$$

Since L^2 is complete, it follows that $\{\widehat{f_j}\}$ has a limit when $j \to \infty$, which we call $\widehat{f}$.

It also follows that

$$\sum_{i=1}^n \int |\partial_i \phi(x)|^2\,dx = \frac{1}{(2\pi)^n}\sum_{i=1}^n \int |\xi_i \hat{\phi}(\xi)|^2\,d\xi = \frac{1}{(2\pi)^n}\int |\xi|^2 |\hat{\phi}(\xi)|^2\,d\xi$$

It is now natural to define the Sobolev norms

$$\|\phi\|_{H^s} = \sqrt{\int (1 + |\xi|^2)^s |\hat{\phi}(\xi)|^2\,d\xi} \tag{4.13}$$

For integer values of s this corresponds to L^2 norms of derivatives of ϕ, but the norm makes sense and is useful also for real s. This shows that there is a relation between decay of the Fourier transform and regularity of the function.

Problem 4.1 Find the Fourier transform of $e^{-|x|}$, $x \in \mathbf{R}$.
Problem 4.2 Find the inverse Fourier transform of $\sin|\xi|/|\xi|$, $\xi \in \mathbf{R}$.