# MATH 168A - SAMPLING AND NYQUIST'S THEOREM

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## 8.1

Throughout the course so far, we have dealt almost exclusively with functions $f(x)$ of a real variable $x$. Most of the integrals we have evaluated are on the entire real line. While this all works out fine theoretically, we cannot in practice hope to make such calculations. The reality is that we can only really evaluate a function at a finite set of points. Doing so is called *sampling*.

The natural question to ask is "How much information about my original function will I able to deduce from a given set of samples"? A rather shocking answer to this question was given by Nyquist, Shannon, and others in the mid 1900s. In order to properly formulate their results, we need only a couple definitions.

### 8.1.1.

**Definition.** Let $f$ be a function defined on $\mathbb{R}$. If the fourier transform $\hat{f}$ happens to have bounded support, say in $[-L, L]$, we say that $f$ is *bandlimited* (or $L$*-bandlimited*). The *bandwidth* of $f$ is the size of the support of $\hat{f}$ ($2L$ in this case).

*Remark.* An $L$-bandlimited function in either $L^1$ or $L^2$ is always infinitely differentiable. To see this, we note that $\hat{f}$ is in $L^1$ and use the Fourier inversion theorem to write

$$f(x) = \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(\xi) e^{ix\xi} \, d\xi \tag{1}$$

Then since $\hat{f}$ has finite support, we can differentiate under the integral sign as many times as we want.

Now we can state and prove the theorem:

**Theorem** (Nyquist's Theorem)**.** *If $f \in L^2(\mathbb{R})$ and $f$ is $L$-bandlimited, then $f$ is completely determined by the samples $\{f(\frac{n\pi}{L})\}_{n\in\mathbb{Z}}$.*

*Proof.* By (1), we can think of the sequence $\{2\pi f(\frac{n\pi}{L})\}_{n\in\mathbb{Z}}$ as the Fourier coefficients of $\hat{f}$. Then by the inversion theorem for $L^2$ functions, we have for $|\xi| < L$

$$\hat{f}(\xi) = \frac{\pi}{L} \operatorname{LIM}_N \sum_{-N}^{N} f\left(\frac{n\pi}{L}\right) e^{\frac{-i\pi n\xi}{L}} \tag{2}$$

where LIM denotes that we are taking an $L^2$ limit. We will now refer to the right hand side of (2) simply as $\frac{\pi}{L}\sum_{-\infty}^{\infty} f(\frac{n\pi}{L})e^{\frac{-i\pi n\xi}{L}}$. However, this last function is periodic with period $2L$, so we have determined completely the Fourier transform $\hat{f}$. Since $f$ is in $L^2$, this means that we have completely determined $f$. $\square$

*Note.* The complex exponentials $e^{\pm iLx}$ have frequency $\frac{L}{2\pi}$. If $f$ is $L$-bandlimited, the theorem tells us that we have to sample at a frequency of $\frac{L}{\pi}$ if we hope to recover $f$ completely. Note that this is twice the highest frequency appearing in the Fourier expansion of an $L$-bandlimited function. This optimal rate is referred to as the *Nyquist rate*.

This theorem is certainly very striking, but to get full value out of it we would like to explicitly reconstruct $f$ from the samples. We will be able to do this with an additional assumption about the rate of decay of $f$: Suppose $f$ is such that

$$\sum_{n=-\infty}^{\infty} \left| f\left(\frac{n\pi}{L}\right) \right| < \infty$$

Plugging in our formula for $\hat{f}$ into (1) gives us that

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi_{[-L,L]}(\xi) \hat{f}(\xi) e^{ix\xi}\, d\xi \\ &= \frac{1}{2\pi} \frac{\pi}{L} \int_{-L}^{L} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) e^{ix\xi - \frac{i\pi n\xi}{L}}\, d\xi \\ &= \frac{1}{2L} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \int_{-L}^{L} e^{ix\xi - \frac{i\pi n\xi}{L}}\, d\xi \\ &= \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \operatorname{sinc}(Lx - n\pi) \end{aligned} \tag{3}$$

**8.1.2.** The interpolation formula (3) contains an infinite sum and is thus only really of theoretical significance. In practice, we would have to approximate it by a finite sum. The problem with this is that $\operatorname{sinc}(Lx - n\pi) \sim \frac{1}{n}$, and since $\sum_{n=1}^{\infty} n^{-1}$ does not converge, we would expect the partial sums of (3) to converge very slowly.

To find an interpolation formula that converges faster, we can *oversample*. For instance, suppose that $f$ is an $(L-\eta)$-bandlimited function for some $\eta > 0$. Now choose a function $\phi$ such that $\hat{\phi}(\xi) = 1$ if $|\xi| \leq L - \eta$ and $\hat{\phi} = 0$ if $|\xi| > L$. We refer to $\phi$ as a *window function*.

From (2) it follows that

$$\hat{f}(\xi) = \frac{\pi}{L} \sum_{-\infty}^{\infty} f(\frac{n\pi}{L}) e^{\frac{-i\pi n\xi}{L}} \quad \text{if } |\xi| \leq L$$

By construction of $\phi$, we can write $\hat{f}(\xi) = \hat{f}(\xi)\hat{\phi}(\xi)$. Calculating as before we get

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)\hat{\phi}(\xi) e^{ix\xi}\, d\xi \\ &= \frac{1}{2\pi} \frac{\pi}{L} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \int_{-L}^{L} \hat{\phi}(\xi) e^{ix\xi - \frac{i\pi n\xi}{L}}\, d\xi \\ &= \frac{1}{2L} \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \phi(x - \frac{n\pi}{L}) \end{aligned} \tag{4}$$

*Note.* Our previous formula comes from choosing $\hat{\phi} = \chi_{[-L,L]}$.

The general rule is that the smoother $\hat{\phi}$ is, the more rapidly the interpolation for $f$ will converge. The costs of choosing a good $\hat{\phi}$ are that it may be difficult to compute $\phi$ from the Fourier inversion formula and that we have to sample above the Nyquist rate (since $f$ is $(L-\eta)$-bandlimited, the Nyquist rate would have been $\frac{\pi}{L-\eta}$, not $\frac{\pi}{L}$).