MATH 168A - PERIODIC FUNCTIONS AND THE FINITE FOURIER TRANSFORM

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If we were to sample a periodic function, we would find that we need only a finite number of samples since the function eventually repeats itself. In this case, there is a finite analogue to the Fourier series that will be useful in studying periodic functions. First, we define the *finite Fourier transform.*

Definition. Suppose $\{x_0, \ldots, x_{m-1}\}$ are complex numbers. The *finite Fourier transform* of this sequence is the sequence $\{\hat{x}_1, \ldots, \hat{x}_{m-1}\}$, where

$$\hat{x}_k = \frac{1}{m} \sum_{j=0}^{m-1} x_j e^{-\frac{2\pi ijk}{m}} \tag{1}$$

A calculation will show (see homework) that the inverse of the finite Fourier transform is given by

$$x_j = \sum_{k=0}^{m-1} \hat{x}_k e^{\frac{2\pi ijk}{m}} \tag{2}$$

Sometimes it will be useful for us to think of a function defined on $[0, L]$ as an L-periodic function, in which case we can consider $\{x_j\}$ and $\{\hat{x}_k\}$ as bi-infinite sequences such that $x_{j+m} = x_j$, etc. These are referred to as m-periodic sequences.

Now we will see what we can do with periodic functions. Suppose that f is an L-periodic function with Fourier coefficients $\{\hat{f}(n)\}$.

Definition. A periodic function f is called N-bandlimited if $\hat{f}(n) = 0$ for $|n| \geq N$.

For such a function, the inversion theorem for Fourier series gives us

$$f(x) = \frac{1}{L} \sum_{n=1-N}^{N-1} \hat{f}(n) e^{\frac{2\pi inx}{L}} \tag{3}$$

Suppose now that we sample f at the points $\frac{jL}{2N-1}$ for $j = 0, 1, \ldots, 2N-2$. Using (3) we obtain

$$\begin{aligned}\sum_{j=0}^{2N-2} f(\tfrac{jL}{2N-1})e^{-\frac{2\pi ikj}{2N-1}} &= \sum_{j=0}^{2N-2} \frac{1}{L} \sum_{n=1-N}^{N-1} \hat{f}(n) e^{\frac{2\pi injL}{(2N-1)L} - \frac{2\pi ikj}{2N-1}} \\ &= \frac{1}{L} \sum_{n=1-N}^{N-1} \hat{f}(n) \sum_{j=0}^{2N-2} e^{\frac{2\pi ij}{2N-1}(n-k)} \\ &= \frac{2N-1}{L} \hat{f}(k)\end{aligned}$$

In this calculation we have made use of the formula

$$\sum_{j=0}^{m-1} e^{\frac{2\pi ij}{m}(k-l)} = \begin{cases} m & k = l \\ 0 & k \neq l \end{cases}$$

This shows that if f is N-bandlimited, then we can compute the Fourier coefficients $\hat{f}(k)$ for $k \in \{1-N, \ldots, N-1\}$ by computing the finite Fourier transform of the sequence of samples $\{f(0), \ldots, f(\frac{(2N-2)L}{2N-1})\}$.

We thus have a periodic version of Nyquist's Theorem

Theorem (Nyquist's Theorem for Periodic Functions)**.** *If f is an L-periodic function and $\hat{f}(n) = 0$ for $|n| \geq N$, then f can be reconstructed from the equally spaced samples $\{f(\frac{jL}{2N-1})\}$, $j = 0, \ldots, 2N-2$.*

Like before, we can derive a nice interpolation formula that gives f specifically in terms of the samples:

$$\begin{aligned} f(x) = \frac{1}{L} \sum_{n=1-N}^{N-1} \hat{f}(n) e^{\frac{2\pi inx}{L}} &= \frac{1}{L} \sum_{n=1-N}^{N-1} \frac{L}{2N-1} \sum_{j=0}^{2N-2} f(\tfrac{jL}{2N-1}) e^{-\frac{2\pi inj}{2N-1}} e^{\frac{2\pi inx}{L}} \\ &= \frac{1}{2N-1} \sum_{j=0}^{2N-2} f(\tfrac{jL}{2N-1}) \sum_{n=1-N}^{N-1} e^{-\frac{2\pi inj}{2N-1}} e^{\frac{2\pi inx}{L}} \\ &= \frac{1}{2N-1} \sum_{j=0}^{2N-2} f(\tfrac{jL}{2N-1}) \frac{\sin[\pi(2N-1)(\frac{x}{L} - \frac{j}{2N-1})]}{\sin[\pi(\frac{x}{L} - \frac{j}{2N-1})]} \end{aligned} \tag{4}$$

Even if f is not bandlimited, (4) still defines an N-bandlimited function that agrees with f at the sample points. Again, we will have aliasing distortion: high frequency data in f will mess with the low frequencies in (4). If f is discontinuous, (4) will also display Gibbs oscillations.