Math 168A Practice Midterm Fall 09

1. Find the Radon transform of $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}\chi_{B_R}(x,y)$, where

$$B_R = B_R(0) = \{(x,y) \mid x^2 + y^2 < R^2\}$$

2. Find the Fourier transform of the characteristic function $\chi_{[-1,1]}(x) = 1$, when $|x| < 1$ and 0 otherwise.

3. Find the Hilbert transform of the characteristic function $\chi_{[-1,1]}(x) = 1$, when $|x| < 1$ and 0 otherwise.

4. Suppose f and g are continuous functions with compact support. Prove that $\mathcal{H}(f*g) = (\mathcal{H}f)*g = f*\mathcal{H}g$. ($\mathcal{H}$ is the Hilbert transform).

5. Let A_θ denote rotation through an angle θ. Set $\omega(\theta) = (\cos\theta, \sin\theta)$ and let $f_\theta(x,y) = f(A_\theta(x,y))$, where A_θ is the matrix rotating (x,y) an angle θ.
   (a) Show that $\mathcal{R}f_\theta(t,\omega(\phi)) = \mathcal{R}f(t,\omega(\phi+\theta))$.
   (b) Use (a) to show that $\partial_\theta \mathcal{R}f(t,\omega(\theta)) = \mathcal{R}[\Omega f](t,\omega(\theta))$ where $\Omega f = (y\partial_x - x\partial_y)f$.

6. The function $\text{sgn}(x) = 1$, when $x > 0$ and $\text{sgn}(x) = -1$, when $x \leq 0$ is not in L^1. The Fourier transform can therefore not be obtained directly from $\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)e^{-ix\xi}\,dx$. Still it can be defined as distribution limit as $\varepsilon \to 0$ of $\hat{f}_\varepsilon(\xi) = \int_{-\infty}^{+\infty} f_\varepsilon(x)e^{-ix\xi}\,dx$, where $f_\varepsilon(x) = f(x)e^{-\varepsilon|x|}$. Show that the limit distribution exist

$$\int \widehat{\text{sgn}}(\xi)\phi(\xi)\,d\xi = \lim_{\varepsilon\to 0}\int \widehat{\text{sgn}}_\varepsilon(\xi)\phi(\xi)\,d\xi$$

   for all rapidly decreasing smooth test functions ϕ, and determine what is.