

Math 168A Practice Exam Solutions

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1. By the formula for the Radon transform of radially symmetric functions, we have

$$\mathfrak{R}f(t, \omega) = 2 \int_{|t|}^{\infty} \frac{r \chi_{B_R}(r)}{r \sqrt{r^2 - t^2}} dr = 2 \int_{|t|}^R \frac{1}{\sqrt{r^2 - t^2}} dr = 2 \log \left| r + \sqrt{r^2 - t^2} \right|_{|t|}^R$$

so that

$$\mathfrak{R}f(t, \omega) = \begin{cases} 2 \log \left| \frac{R + \sqrt{R^2 - t^2}}{t} \right| & |t| \leq R \\ 0 & |t| > R \end{cases}$$

- 2.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \chi_{[-1,1]}(x) e^{-ix\xi} dx = -\frac{1}{i\xi} e^{-ix\xi} \Big|_{x=-1}^{x=1} = \frac{e^{i\xi} - e^{-i\xi}}{i\xi} = \frac{2 \sin \xi}{\xi}$$

3. We compute this as

$$\frac{i}{\pi} \text{PV}(\chi_{[-1,1]}(s) * \frac{1}{s}) = \frac{i}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|x| > \epsilon} \frac{\chi_{[-1,1]}(s)}{t - s} ds = \frac{i}{\pi} \lim_{\epsilon \rightarrow 0} \int_{1 \geq |x| > \epsilon} \frac{ds}{t - s}$$

If we write $\int_{1 \geq |x| > \epsilon}$ as $\int_{-1}^{-\epsilon} + \int_{\epsilon}^1$ and then change variables to $u = -s$ in the first integral, we get

$$\frac{i}{\pi} \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^1 \left[\frac{1}{t - s} + \frac{1}{t + s} \right] ds = \frac{i}{\pi} \lim_{\epsilon \rightarrow 0} \left[\log \left| \frac{t + s}{t - s} \right| - \log \left| \frac{t + \epsilon}{t - \epsilon} \right| \right] = \frac{i}{\pi} \log \left| \frac{t + 1}{t - 1} \right|$$

4. With the given assumptions on f and g , we can say that $f, g \in L^1 \cap L^2$. This allows us to simply write

$$\mathcal{H}(f * g) = \mathcal{F}^{-1}(\text{sgn} \widehat{f * g}) = \mathcal{F}^{-1}(\text{sgn} \hat{f} \hat{g}) = \mathcal{F}^{-1}(\text{sgn} \hat{f}) * g = (\mathcal{H}f) * g$$

The other equality is similar.

5. (a) The relevant observation here is that $A_{\theta} \omega(\phi) = \omega(\phi + \theta)$. Then we calculate

$$\begin{aligned} \mathfrak{R}f_{\theta}(t, \omega(\phi)) &= \int_{-\infty}^{\infty} f_{\theta}(s \hat{\omega}(\phi) + t \omega(\phi)) ds = \int_{-\infty}^{\infty} f(s A_{\theta} \hat{\omega}(\phi) + t A_{\theta} \omega(\phi)) ds \\ &= \int_{-\infty}^{\infty} f(s \hat{\omega}(\phi + \theta) + t \omega(\phi + \theta)) ds \\ &= \mathfrak{R}f(t, \omega(\phi + \theta)) \end{aligned}$$

(b) By (a) we can write

$$\begin{aligned}\partial_\theta \mathfrak{R}f(t, \omega(\theta)) &= \lim_{\phi \rightarrow 0} \frac{\mathfrak{R}f(t, \omega(\theta + \phi)) - \mathfrak{R}f(t, \omega(\theta))}{\phi} \\ &= \lim_{\phi \rightarrow 0} \frac{\mathfrak{R}f_\phi(t, \omega(\theta)) - \mathfrak{R}f(t, \omega(\theta))}{\phi} \\ &= \lim_{\phi \rightarrow 0} \mathfrak{R} \left[\frac{f_\phi - f}{\phi} \right] (t, \omega(\theta))\end{aligned}$$

If we show that $\frac{f_\phi - f}{\phi}$ converges to $y\partial_x f - x\partial_y f$ as $\phi \rightarrow 0$, then we can use the Dominated Convergence theorem to pass the limit through the Radon transform to obtain the result.

Sure enough, we can write A_ϕ in matrix form as $A_\phi = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix}$ and let $(x(\phi), y(\phi)) = A_\phi(x, y) = (x \cos \phi + y \sin \phi, -x \sin \phi + y \cos \phi)$. Then

$$\lim_{\phi \rightarrow 0} \frac{f_\phi(x, y) - f(x, y)}{\phi} = \partial_\phi f(x(\phi), y(\phi)) \Big|_{\phi=0} = y\partial_x f(x, y) - x\partial_y f(x, y)$$

so we are done.

6. First we compute

$$\hat{f}_\epsilon(\xi) = \int_0^\infty e^{-\epsilon x} e^{-ix\xi} dx - \int_{-\infty}^0 e^{\epsilon x} e^{-ix\xi} dx = \frac{2\xi}{i(\xi^2 + \epsilon^2)}$$

Thus, we have that for any $\phi \in C_0^\infty(\mathbb{R})$,

$$\hat{f}(\phi) = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^\infty \frac{2x\phi(x)}{i(x^2 + \epsilon^2)} dx$$

Now the trick is to manipulate the integrand so that we can take the limit inside the integral. We will do this by noting that (since ϕ has compact support, say contained in $[-R, R]$)

$$\int_{-\infty}^\infty \frac{2x\phi(x)}{i(x^2 + \epsilon^2)} dx = \int_{-R}^R \frac{2x\phi(x)}{i(x^2 + \epsilon^2)} - \frac{2x\phi(0)}{i(x^2 + \epsilon^2)} dx = \int_{-R}^R \frac{2}{i} \left[\frac{\phi(x) - \phi(0)}{x} \right] \frac{x^2}{(x^2 + \epsilon^2)} dx$$

Now $\frac{\phi(x) - \phi(0)}{x}$ is a smooth function with compact support that we will denote by $\psi(x)$ (in fact we can write the formula $\psi(x) = \frac{1}{x} \int_0^x \phi'(t) dt = \int_0^1 \phi'(xs) ds$). Since the rest of the integrand is bounded, we can pass our limit through and obtain

$$\hat{f}(\phi) = \frac{2}{i} \int_{-R}^R \frac{\phi(x) - \phi(0)}{x} dx$$

Thus we have shown

$$\hat{f} = \frac{2}{i} \text{PV}\left(\frac{1}{x}\right)$$