Math 21C Final Exam Solutions, Lindblad, Fall 02

1. (a). $\mathbf{r}(1) = \mathbf{i} + \mathbf{k}$. $\mathbf{r}'(t) = 2t\mathbf{i} - 2t\mathbf{j} + 3t^2\mathbf{k}$. $\mathbf{r}'(1) = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. Tangent line given by $\mathbf{r}_0(t) = (1+2t)\mathbf{i} + (-2t)\mathbf{j} + (1+3t)\mathbf{k}$.
(b).

$$\int_0^1 \sqrt{4t^2 + 4t^2 + 9t^4}\, dt \;=\; \int_0^1 t\sqrt{8 + 9t^2}\, dt \;=\; \frac{1}{27}(8+9t^2)^{3/2}\Big|_0^1 = \frac{17^{3/2} - 4}{27}.$$

2. $\overrightarrow{PQ} = \langle 0, 1, -1\rangle$, $\overrightarrow{PR} = \langle 2, 0, -1\rangle$ Normal to the plane is

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ 2 & 0 & -1 \end{vmatrix} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

Equation of plane is $-(x-0) - 2(y-1) - 2(z-1) = 0$ which can be written as $x + 2y + 2z = 4$.
(b). area is $|\overrightarrow{PQ} \times \overrightarrow{PR}|/2 = \sqrt{(-1)^2 + (-2)^2 + (-2)^2}/2 = 3/2$.
(c).

$$\cos\angle P \;=\; \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} \;=\; \frac{\langle 0,1,-1\rangle\cdot\langle 2,0,-1\rangle}{|\langle 0,1,-1\rangle||\langle 2,0,-1\rangle|} \;=\; \frac{1}{\sqrt{2}\sqrt{5}}.$$

$$\cos\angle Q \;=\; \frac{\overrightarrow{QP}\cdot\overrightarrow{QR}}{|\overrightarrow{QP}||\overrightarrow{QR}|} \;=\; \frac{\langle 0,-1,1\rangle\cdot\langle 2,-1,0\rangle}{|\langle 0,-1,1\rangle||\langle 2,-1,0\rangle|} \;=\; \frac{1}{\sqrt{2}\sqrt{5}}.$$

$$\cos\angle R \;=\; \frac{\overrightarrow{RP}\cdot\overrightarrow{RQ}}{|\overrightarrow{RP}||\overrightarrow{RQ}|} \;=\; \frac{\langle -2,0,1\rangle\cdot\langle -2,1,0\rangle}{|\langle -2,0,1\rangle||\langle -2,1,0\rangle|} \;=\; \frac{4}{5}.$$

3.

$$\nabla T = \left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$\nabla T(1,2,2) \;=\; (1,2,2)/27.$$

Direction from $(1,2,2)$ to $(2,1,3)$ is $\langle 1,-1,1\rangle$. Unit vector in this direction is $\mathbf{u} = \langle 1,-1,1\rangle/\sqrt{3}$,

$$D_{\mathbf{u}}T(1,2,2) \;=\; \nabla T(1,2,2)\cdot\mathbf{u} = \frac{1}{27\sqrt{3}}.$$

(b). The direction of greatest increase of T is ∇T which is a positive multiple of $-\langle x,y,z\rangle$. But the vector $-\langle x,y,z\rangle$ is the vector from (x,y,z) to the origin $(0,0,0)$.

4. (a). Two planes are parallel if and only if their normals are parallel. The normal to a level surface $f(x,y,z)$ =constant is ∇f. Hence the normal to the ellipsoid at the point (x,y,z) is $\langle 8x, 2y, 2z\rangle$. The plane $x+2y-z=0$ has normal $\langle 1,2,-1\rangle$. Hence the tangent plane to the ellipsoid is parallel to the plane $x+2y-z=0$ if and only if the vectors $\langle 8x,2y,2z\rangle$ and $\langle 1,2,-1\rangle$ are parallel, that is for some value of λ,

$$8x=\lambda, \qquad 2y=2\lambda, \qquad 2z=-\lambda.$$

This gives $(x,y,z)=(\lambda/8,\ \lambda,\ -\lambda/2)$. Plugging in to the equation of the ellipsoid we get

$$4 \;=\; 4(\lambda/8)^2 \;+\; \lambda^2 \;+\; (\lambda/2)^2 \;=\; \frac{21}{16}\lambda^2,$$

so $\lambda=\pm 8/\sqrt{21}$, and the points are $(1/\sqrt{21},\ 8/\sqrt{21},\ -4/\sqrt{21})$ and $(-1/\sqrt{21},\ -8/\sqrt{21},\ 4/\sqrt{21})$

(b). Since the normal is $\langle 1,2,-1\rangle$, the equation of the tangent plane is $x+2y-z$ =constant. We evaluate the constant by plugging in the points we just found and get tangent plane $x+2y-z=\sqrt{21}$ at the first point and $x+2y-z=-\sqrt{21}$ at the second point.

5. (a). $\nabla f=\langle 2x+3y, 10y+3x\rangle=\langle 0,0\rangle$ if and only if $(x,y)=(0,0)$. This is the only critical point.

$$D \;=\; \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} \;=\; \begin{vmatrix} 2 & 3 \\ 3 & 10 \end{vmatrix} \;=\; 11>0,$$

since $f_{xx}>0$, the point $(0,0)$ is a local minimum.

(b). Using Lagrange multipliers we get the equations

$$\begin{cases} 2x+3y=\lambda 2x \\ 10y+3x=\lambda 2y \\ x^2+y^2=4. \end{cases}$$

We can eliminate λ by multiplying the first equation by y and the second equation by x and subtracting, to get $3y^2+2xy-10xy-3x^2=0$ which becomes $3y^2-8xy-3x^2=0$. This factors as $(3y+x)(y-3x)=0$, so $y=3x$ or $x=-3y$. In the first case we get $x^2+(3x)^2=4$ so $(x,y)=(\sqrt{2/5}, 3\sqrt{2/5})$ and $f=126/5$, and in the second case we get $(-3y)^2+y^2=4$ so $(x,y)=(-3\sqrt{2/5}, \sqrt{2/5}$ and $f=90/5$. The maximum value of f on the circle is $126/5$ and the minimum value is $90/5$.

(c). The absolute max of f is $126/5$ and the absolute min is $f(0,0)=8$.

6. We will maximize $V=xyz$ over the tetrahedron given by $x\geq 0$, $y\geq 0$, $z\geq 0$, and $2x+2y+z\leq 120$. We first look for interior critical points and get $\nabla V=\langle yz, xz, xy\rangle=\langle 0,0,0\rangle$ which implies that at least two of x,y,z vanish, so (x,y,z) is on the boundary of the tetrahedron where $V=0$. The boundary of the tetrahedron consists of four triangles, three on the coordinate planes where $V=0$, and one

on the plane $2x + 2y + z = 120$ with $x \geq 0$, $y \geq 0$ and $z \geq 0$. The edges of this triangle lie in the coordinate planes and so $V = 0$ on the edges. It is now clear that the maximum value of V on the tetrahedron must lie on this triangle $G = 2x + 2y + z = 120$ with $x > 0, y > 0, z > 0$. Using Lagrange multipliers to maximize $V = xyz$ subject to these conditions, we get

$$\begin{cases} yz = 2\lambda \\ xz = 2\lambda \\ xy = \lambda z \\ 2x + 2y + z = 120. \end{cases}$$

Hence

$$yz = xz = 2xy$$

Since x, y and z are positive, we can divide to get $y = x$ and $z = 2x$. Solving the constraint gives $2x + 2x + 2x = 120$ so $x = y = 20$ and $z = 40$. Then $V = 20^2(40) = 16000$ cubic inches.

7. (a). Set

$$D \ = \ \{(x,y) : 0 \leq y \leq 1, \quad 1 \leq x \leq 1/y\}.$$

The integral equals $\displaystyle\iint_D x^2 e^{-x^2}\, dA$.

(b). After sketching the domain, we can write it as

$$D \ = \{(x,y) : 1 \leq x \leq \infty, \quad 0 \leq y \leq 1/x\}.$$

Then

$$\iint_D x^2 e^{-x^2}\, dA \ = \ \int_1^\infty \int_0^{1/x} x^2 e^{-x^2}\, dxdy \ = \ \int_1^\infty x^2 e^{-x^2} y\Big|_0^{1/x}\, dx$$
$$= \ \int_1^\infty x e^{-x^2/2}\, dx \ = \ -\frac{e^{-x^2}}{2}\Bigg|_1^\infty \ = \ \frac{e}{2}.$$

8. Notice that

$$\iint_R (1+y)\cos(x^2+y^2)\, dA = \iint_R \cos(x^2+y^2)\, dA \ + \ \iint_R y\cos(x^2+y^2)\, dA,$$

and the second integral vanishes by symmetry. This is helpful, but we don't need it. Using polar coordinates. The integral we want to compute is

$$\int_0^{2\pi}\int_1^2 (1 + r\sin\theta)\ \cos(r^2)\, r\, drd\theta \ = \ \int_0^{2\pi}\int_1^2 r\cos(r^2) \ + \ \sin\theta r^2\cos(r^2)\, drd\theta.$$

We cannot integrate this, so we switch the order of integration to get

$$\int_1^2 \int_0^{2\pi} r\cos(r^2) + \cos\theta\, r^2\cos(r^2)\, d\theta dr \;=\; \int_1^2 \left(r\cos(r^2)\theta - \cos\theta\, r^2\cos(r^2)\right)\Big|_0^{2\pi} dr$$

$$= \int_1^2 2\pi r\cos r^2\, dr \;=\; \pi\sin(r^2)\big|_1^2 = \pi(\sin 4 - \sin 1).$$

9. The surface is $z = 8 - x - 2y$ and its area is

$$\iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \;=\; \iint_D \sqrt{1 + (-1)^2 + (-2)^2}\, dA \;=\; \iint_D \sqrt{6}\, dA$$

$$= \int_0^1 \int_{2x^2}^{1+x^2} \sqrt{6}\, dy dx \;=\; \sqrt{6}\int_0^1 1 + x^2 - 2x^2\, dx \;=\; \sqrt{6}(x - x^3/3)\big|_0^1 = \frac{2\sqrt{6}}{3}$$

10. Describing D as a type I region we get $0 \leq y \leq 1,\; 2x \leq y \leq 4 - 2x$. Then the volume of E is

$$\iint_D x - y + 20\, dA \;=\; \int_0^1 \int_{2x}^{4-2x} x + 20 - y\, dy dx$$

$$= \int_0^1 (x+20)y - y^2/2\big|_{2x}^{4-2x} dx = \int_0^1 (x+20)(4-4x) - (2-x)(4-2x) + 2x^2\, dx$$

$$= \int_0^1 -4x^2 - 76x + 80\, dx \;=\; \frac{-4}{3} - 38 + 80.$$