Review:. 13. Vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$.

Derivative: $\mathbf{r}'(t) = \lim_{h\to 0} \dfrac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \langle f'(t), g'(t), h'(t)\rangle$, is **tangent** to the curve $\mathbf{r}(t)$

Arclength: $L = \int_a^b |\mathbf{r}'(t)|\, dt,$ where $|\mathbf{r}'(t)| = \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}$

14. Functions of several variables $f(x,y)$ and $F(x,y,z)$.
Level curves $f(x,y) = k$ and **level surfaces** $F(x,y,z) = k$.

Partial derivatives:

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h},$$

$$f_y(x,y) = \frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Gradient: $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$, $\nabla F(x,y,z) = \langle F_x(x,y,z), F_y(x,y,z), F_z(x,y,z)\rangle$

Chain Rule case 1: $\dfrac{d}{dt}F(\mathbf{r}(t)) = F_x\dfrac{dx}{dt} + F_y\dfrac{dy}{dt} = \nabla F(\mathbf{r}(t))\cdot \mathbf{r}'(t)$ $\qquad \mathbf{r}(t) = \langle f(t), g(t)\rangle$

Geometrically: The gradient is orthogonal to the tangent line of the level curves.
Directional derivative in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$

$$D_{\mathbf{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+ha, y+hb) - f(x,y)}{h} = f_x(x,y)a + f_y(x,y)b = \nabla f(x,y)\cdot \mathbf{u}$$

Max rate of change is $|\nabla f|$ which occurs in the direction of ∇f.
Chain Rule case 2: If $z = f(x,y)$ where $x = g(s,t)$ and $y = h(s,t)$ then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Tangent plane: The tangent plane to the surface $z = f(x,y)$ at a point (x_0, y_0, z_0) :

(14.8.3) $\qquad z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0), \qquad z_0 = f(x_0, y_0)$

The tangent plane to a level surface $F(x,y,z) = k$ at a point (x_0, y_0, z_0) is

(14.8.4) $F_x(x_0,y_0,z_0)(x - x_0) + F_y(x_0,y_0,z_0)(y - y_0) + F_z(x_0,y_0,z_0)(z - z_0) = 0$

Geometrically: The gradient is orthogonal to the tangent plane of the level surface.
Differentials: If $z = f(x,y)$ and (dx, dy) are variables then the differential of f is

$$dz = f_x(x,y)\, dx + f_y(x,y)\, dy$$

Linear approximation: With $dx = \triangle x$ and $dy = \triangle y$ we have

$$\triangle z = f(x + \triangle x, y + \triangle y) - f(x,y) \sim dz = f_x(x,y)\, \triangle x + f_y(x,y)\, \triangle y$$

Max-min of $f(x,y)$ is a **critical point:** $f_x(x,y) = f_y(x,y) = 0$.
Second derivative test If (x,y) is a critical point and $D = f_{xx}f_{yy} - f_{xy}^2$.
Then it is a local min if $D > 0$ and $f_{xx} > 0$, a local max if $D > 0$ and $f_{xx} < 0$ and saddle point if $D < 0$, i.e. neither max nor min.

Lagrange multipliers. To find the max and min of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$ we find all values of (x,y,z) and λ such that

(14.8.5) $\qquad \nabla f(x,y,z) = \lambda \nabla g(x,y,z), \qquad$ and $\qquad g(x,y,z) = k$

Evaluating $f(x,y,z)$ at all the resulting points gives the max and min.