

Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ be vectors and θ the angle between them.

Length: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Dot product: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}| |\mathbf{b}| \cos \theta$.

\mathbf{a} is parallel to \mathbf{b} if $\mathbf{a} = c\mathbf{b}$, for some scalar c . \mathbf{a} and \mathbf{b} are perpendicular if $\mathbf{a} \cdot \mathbf{b} = 0$.

Vector projection of \mathbf{b} onto \mathbf{a} : $\text{proj}_{\mathbf{a}}(\mathbf{b}) = \text{comp}_{\mathbf{a}}(\mathbf{b})\mathbf{u}$, where $\mathbf{u} = \mathbf{a}/|\mathbf{a}|$ is a unit vector and $\text{comp}_{\mathbf{a}}(\mathbf{b}) = \mathbf{b} \cdot \mathbf{u}$ is the component of \mathbf{b} in the direction of \mathbf{a} .

Cross product: $\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$, if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k},$$

$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$, $\mathbf{a} \times \mathbf{b}$ is perpendicular to both \mathbf{a} and \mathbf{b} .

The area of the triangle with \mathbf{a} and \mathbf{b} as two edges is $A = |\mathbf{a} \times \mathbf{b}|/2$.

The volume of the parallelepiped with \mathbf{a} , \mathbf{b} and \mathbf{c} as three edges is $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

$\mathbf{a} \times \mathbf{b} = 0$ if \mathbf{a} and \mathbf{b} are parallel. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ if \mathbf{a} , \mathbf{b} and \mathbf{c} lie in the same plane.

Parametric equations of a line: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$ where (x_0, y_0, z_0) is on the line and $\langle a, b, c \rangle$ is parallel to it: $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$.

Equations of planes $ax + by + cz + d = 0$, where $\mathbf{n} = \langle a, b, c \rangle$ is a normal to the plane, i.e. it is perpendicular to it. If $P_0(x_0, y_0, z_0)$ is a point in the plane this can also be written as $\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

The distance between a point $P_1(x_1, y_1, z_1)$ and the plane is $D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

Quadratic surfaces: An Ellipsoid $x^2 + y^2 + 2z^2 = 1$, an Elliptic Paraboloid $z = x^2 + y^2$, a Hyperbolic Paraboloid $z = x^2 - y^2$, a Cone $z^2 = x^2 + y^2$ and a Hyperboloid $z^2 = x^2 + y^2 + 1$.

Vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$.

Derivative: $\mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \langle f'(t), g'(t), h'(t) \rangle$, is **tangent** to the curve $\mathbf{r}(t)$

Arclength: $L = \int_a^b |\mathbf{r}'(t)| dt$, where $|\mathbf{r}'(t)| = \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}$

Functions of several variables $f(x, y)$ and $F(x, y, z)$.

Level curves $f(x, y) = k$ and **level surfaces** $F(x, y, z) = k$.

Partial derivatives: $f_x(x, y) = \frac{\partial f}{\partial x}(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$,

$$f_y(x, y) = \frac{\partial f}{\partial y}(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Gradient: $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$, $\nabla F(x, y, z) = \langle F_x(x, y, z), F_y(x, y, z), F_z(x, y, z) \rangle$

Chain Rule case 1: $\frac{d}{dt}F(\mathbf{r}(t)) = F_x \frac{dx}{dt} + F_y \frac{dy}{dt} = \nabla F(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ $\mathbf{r}(t) = \langle f(t), g(t) \rangle$

Geometrically: The gradient is orthogonal to the tangent line of the level curves.

Directional derivative in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$

$$D_{\mathbf{u}}f(x, y) = \lim_{h \rightarrow 0} \frac{f(x+ha, y+hb) - f(x, y)}{h} = f_x(x, y)a + f_y(x, y)b = \nabla f(x, y) \cdot \mathbf{u}$$

Max rate of change is $|\nabla f|$ which occurs in the direction of ∇f .

Chain Rule case 2: If $z = f(x, y)$ where $x = g(s, t)$ and $y = h(s, t)$ then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

Tangent plane: The tangent plane to the surface $z = f(x, y)$ at a point (x_0, y_0, z_0) :

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0), \quad z_0 = f(x_0, y_0)$$

The tangent plane to a level surface $F(x, y, z) = k$ at a point (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Geometrically: The gradient is orthogonal to the tangent plane of the level surface.

Differentials: If $z = f(x, y)$ and (dx, dy) are variables then the differential of f is

$$dz = f_x(x, y) dx + f_y(x, y) dy$$

Linear approximation: With $dx = \Delta x$ and $dy = \Delta y$ we have

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \sim dz = f_x(x, y) \Delta x + f_y(x, y) \Delta y$$

Max-min of $f(x, y)$ is a **critical point:** $f_x(x, y) = f_y(x, y) = 0$.

Second derivative test If (x, y) is a critical point and $D = f_{xx}f_{yy} - f_{xy}^2$.

Then it is a local min if $D > 0$ and $f_{xx} > 0$, a local max if $D > 0$ and $f_{xx} < 0$ and saddle point if $D < 0$, i.e. neither max nor min.

Lagrange multipliers. To find the max and min of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ we find all values of (x, y, z) and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z), \quad \text{and} \quad g(x, y, z) = k$$

Evaluating $f(x, y, z)$ at all the resulting points gives the max and min.

Double integrals. If $R = \{(x, y); a \leq x \leq b, c \leq y \leq d\}$ then:

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx$$

Region of type I: $D = \{(x, y); a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ then

$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

Polar rectangle $R = \{(r, \theta); a \leq r \leq b, \alpha \leq \theta \leq \beta\}$ then

$$\iint_R f(x, y) dA = \int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$

Surface area, $z = f(x, y), (x, y) \in D$:

$$\text{Area}(S) = \iint_D \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2} dA.$$

Triple Integrals If $B = \{(x, y, z); a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$ then

$$\iiint_B f(x, y, z) dV = \int_r^s \int_c^d \int_a^b f(x, y, z) dx dy dz$$

If region of **type 1:** $E = \{(x, y, z); (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$ then

$$\iiint_E f(x, y, z) dV = \iint_D \left(\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right) dA$$